SELF RATIONAL MAPS OF $K3$ SURFACES

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Abstract. We prove that a very general projective $K3$ surface does not admit a dominant self rational map of degree at least two.

1. Introduction

The purpose of this paper is to prove the following conjecture:

Theorem 1.1. *There is no dominant self rational map $\phi : X \dashrightarrow X$ of degree $\deg\phi > 1$ for a very general projective $K3$ surface X of genus $g \geq 2$.*

For the background of this conjecture, please also see [D].

Self rational maps of $K3$ surfaces arise naturally in several contexts. There are special $K3$ surfaces with nontrivial self rational maps. Here are two typical examples [D]:

- if X is an elliptic $K3$ surface, i.e., a $K3$ surface admitting an elliptic fiberation $X/\mathbb{P}^1$, there are self rational maps $\phi : X \dashrightarrow X$ of degree $\deg\phi > 1$ mapping $X/\mathbb{P}^1$ to $X/\mathbb{P}^1$ fiberwisely;
- if X is a Kummer surface, i.e., a $K3$ surface birational to the quotient of an abelian surface by an involution, there are self rational maps $\phi : X \dashrightarrow X$ of $\deg\phi > 1$ descended from the abelian surface.

To our knowledge, these are the only special $K3$ surfaces known to have nontrivial self rational maps. It would be interesting to find others.

More generally, every variety X birational to a projective family of abelian varieties over some base B admits nontrivial self-rational maps by fixing a multi-section $L \subset X$ of X/B with degree n and sending a point $x \in X_b$ to $L-(n-1)x$. This also works if X/B is birational to a fiberation of quotients of abelian varieties by finite groups.

For a $K3$ surface X over a number field k, the existence of self rational maps for X is closely related to the arithmetic problem on the potential density of k-rational points on X. If there is a rational map $\phi : X \dashrightarrow X$ of $\deg\phi > 1$ over a finite extension $k' \to k$ of the base field, by iterating ϕ, we can produce many k'-rational points on X. Under suitable conditions, these k'-rational points are Zariski dense in X [A-C].

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The existence of self rational maps of a $K3$ surface is also related to its hyperbolic geometry. Algebraic surfaces that are holomorphically dominable by $\mathbb{C}^2$ were classified by G. Buzzard and S. Lu [B-L]. They almost gave a complete answer except for the case of $K3$ surfaces. They showed that elliptic $K3$ and Kummer surfaces are dominable by $\mathbb{C}^2$. However, it is unknown whether a generic $K3$ surface X is dominable by $\mathbb{C}^2$. It is no coincidence that elliptic $K3$ and Kummer surfaces, as the examples of $K3$ surfaces admitting nontrivial self rational maps, are dominated by $\mathbb{C}^2$. Indeed, if there exists a rational map $\phi : X \dashrightarrow X$ with some dilating properties, then by iterating ϕ and taking the limit, we can arrive at a dominating meromorphic map $\mathbb{C}^2 \dashrightarrow X$ [C].

So it becomes a natural question to ask whether a generic $K3$ surface admits a nontrivial self rational map. Here by "generic", we mean "very general", i.e., a $K3$ surface represented by a point in the moduli space of polarized $K3$ surfaces with countably many proper subvarieties removed. Needless to say, the hypothesis of X being very general is necessary since elliptic $K3$'s are parametrized by countably many hypersurfaces in the moduli space. It also means that we have to use this hypothesis in an essential way.

A natural way to prove Theorem 1.1 is via degeneration. Fortunately for us, there are good degenerations of $K3$ surfaces. Every $K3$ surface can be degenerated to a union of rational surfaces. For example, a quartic $K3$ in $\mathbb{P}^3$ can be degenerated to a union of two quadrics or four planes and so on. To see how this can be done in general, we start with a union $W_0 = S_1 \cup S_2$ of two Del Pezzo surfaces meeting transversely along a smooth elliptic curve D (see 2.1 for details). Using the argument in [CLM], we can show that the natural map

$$\begin{aligned}\operatorname{Ext}(\Omega_{W_0}, \mathcal{O}_{W_0}) \to H^0(T^1(X_0)) &= H^0(\mathcal{E}\mathrm{xt}(\Omega_{W_0}, \mathcal{O}_{W_0})) \\ &= H^0(\mathcal{O}_D(-K_{S_1} - K_{S_2}))\end{aligned} \tag{1.1}$$

is surjective. Consequently, a general deformation of W_0 smooths out its singularities along D. And since the dualizing sheaf ω_{W_0} of W_0 is trivial and W_0 is simply connected, W_0 can be deformed to a complex $K3$ surface (not necessarily projective). If we further assume that W_0 possesses an indivisible ample line bundle L with $L^2 = 2g - 2$, then we can deform W_0 while "preserving" L and thus deform W_0 to a smooth projective $K3$ surface of genus g. Since the moduli space of polarized $K3$ surfaces with a fixed genus g is irreducible, this argument shows that every polarized $K3$ surface (S, L) can be degenerated to $(S_1 \cup S_2, L)$ described as above.

Note that W_0 is constructed by gluing S_1 and S_2 transversely along D via two immersions $i_k : D \hookrightarrow S_k$ for $k = 1, 2$. A line bundle L on W_0 is given by two line bundles $L_k \in \operatorname{Pic}(S_k)$ such that L_1 and L_2 agrees on D, i.e.,

$$i_1^* L_1 = i_2^* L_2. \tag{1.2}$$

Naturally, we expect that the existence of self rational maps for generic $K3$ surfaces will induce a self rational map for such a union $S_1 \cup S_2$. To be more precise, we let W/Δ be a family of $K3$ surfaces of genus g over the disk $\Delta = \{|t| < 1\}$ whose central fiber W_0 is a union $S_1 \cup S_2$ given as above. It turns out that W is singular and we need to work with a resolution X of singularities of W (see 2.3). Suppose that there are rational maps $\phi_t : X_t \dashrightarrow X_t$ for all $t \neq 0$. It is easy to see that ϕ_t can be extended to a rational map $\phi : X \dashrightarrow X$ after a base change. Basically, we are trying to study the self-rational maps ϕ_t by studying ϕ_0. However, the rational map $\phi_0 : S_1 \cup S_2 \dashrightarrow S_1 \cup S_2$ does not tell us much itself because, among other things,

- S_k might very well be contracted by ϕ, although it does not turn out to be the case (see Proposition 2.6);
- ϕ might not be regular along $D = S_1 \cap S_2$, i.e., D is contained in the indeterminate locus of ϕ.

To really understand the rational map ϕ, we need to resolve the indeterminacy of ϕ first. Namely, there exists a birational regular map $f : Y \to X$ such that $\varphi = \phi \circ f$ is regular with the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[d, "f"'] & X \\ X \arrow[ur, dashed, "\phi"'] & \end{tikzcd} \tag{1.3}$$

We can make Y_0 as "nice" as possible by the stable reduction theorem in [KKMS], although that comes at a cost that we may get many "irrelevant" components of Y_0 that are contracted by φ. By adjunction and Riemann-Hurwitz, we can figure out all the "relevant" components of Y_0, which turn out to be the union $\mathcal{S}$ of the components of $E \subset Y_0$ with discrepancy $a(E, X) = 0$ under f (see 2.5 for details). This occupies the first part of our proof.

We construct Y as a resolution of indeterminacy of ϕ. Alternatively and equivalently, we can also construct Y as follows. Fixing a sufficiently ample divisor $\mathcal{L}$ on X, we can construct Y by resolving the base locus $\mathrm{Bs}(f_*\varphi^*\mathcal{L})$ of the linear series $f_*\varphi^*\mathcal{L}$ as $\mathcal{L}$ varies in $|\mathcal{L}|$. It is not hard to see that $\mathrm{Bs}(f_*\varphi^*\mathcal{L})$, which is merely the indeterminacy of ϕ, is independent of our choice of Y and $\mathcal{L}$.

It turns out that

$$\operatorname{supp} f_*(\varphi^*\mathcal{L} \cap \mathcal{T}) \subset \mathrm{Bs}(f_*\varphi^*\mathcal{L}_0) \tag{1.4}$$

where $\mathcal{T} = Y_0 - \mathcal{S}$. A large part of this paper is devoted to the study of the curve $f_*(\varphi^*\mathcal{L} \cap \mathcal{T})$. As $\mathrm{Bs}(f_*\varphi^*\mathcal{L})$, this curve does not depend on our choice of Y and $\mathcal{L}$. It is "rigid" in the sense that it has only countably many possible configurations. More precisely, it is contained in a union Σ of countably many rational curves on X_0. This union Σ is determined completely by the Kodaira-Spencer class of W. Indeed, it is determined by the T^1 class of W,

i.e., the singularities of W lying on D. In particular, it does not depend on ϕ. So by iterating ϕ, we can show that some components of Σ are contracted or mapped onto some other components of Σ, which leads to a proof of the main theorem.

One of the crucial facts employed in our proof is

$$\mathrm{End}(D) = \mathbb{Z} \tag{1.5}$$

where $\mathrm{End}(D)$ is the ring of the endomorphisms of D with a fixed point. This holds because D is an elliptic curve of general moduli and hence carries no complex multiplication. In some sense, the triviality of self rational maps of a general $K3$ surface comes down to the triviality of self rational maps of a general elliptic curve. Since elliptic curves are customarily regarded as Calabi-Yau (CY) manifolds of dimension one, this suggests that the same holds in higher dimension and there might be a way to prove it inductively, at least for CY manifolds which are complete intersections in $\mathbb{P}^n$. We will propose the following conjecture for quintic threefolds but say no more.

Conjecture 1.2. *A very general quintic threefold X in $\mathbb{P}^4$ does not admit a self rational map $\phi : X \dashrightarrow X$ of degree $\deg \phi > 1$.*

Conventions. We work exclusively over $\mathbb{C}$ and with analytic topology wherever possible. Clearly, Theorem 1.1 fails trivially in positive characteristic. A $K3$ surface in this paper, unless specified otherwise, is always projective.

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2. Degeneration of $K3$ Surfaces and Resolution of Indeterminacy

2.1. **Degeneration of $K3$ surfaces.** Let W/Δ be a family of $K3$ surfaces over the disk Δ whose general fibers W_t are general $K3$ surfaces of genus $g \geq 2$ and whose central fiber is a union of two smooth rational surfaces $W_0 = S_1 \cup S_2$ meeting transversely along a smooth elliptic curve $D = S_1 \cap S_2$. And there is an indivisible line bundle L on W/Δ with $L_t^2 = 2g - 2$ that polarizes W_t for $t \neq 0$.

We choose S_i with the following properties for $i = 1, 2$:

- the anti-canonical divisors $-K_{S_i}$ are ample, i.e., S_i are Del Pezzo surfaces and $D \in |-K_{S_i}|$;
- $K_{S_1}^2 = K_{S_2}^2$;
- there are ample line bundles L_i on S_i such that

$$L_1\Big|_D = L_2\Big|_D, L\Big|_{S_i} = L_i, L_i^2 = g - 1 \text{ and } L_i D = g + 1. \tag{2.1}$$

As outlined in the previous section, we can show that such a union $S_1 \cup S_2$ can be deformed to a $K3$ surface of genus g.

Such surfaces S_1 and S_2 can be chosen in many different ways. We use the degeneration in [CLM] and [Ch]:

- if g is odd, we let $S_i \cong \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and

$$L_i = L\Big|_{S_i} = C_i + \frac{g-1}{2} F_i \tag{2.2}$$

  where C_i and F_i are the generators of $\operatorname{Pic}(S_i)$ with $C_i^2 = F_i^2 = 0$ and $C_i F_i = 1$ for $i = 1, 2$;
- if $g \geq 4$ is even, we let $S_i \cong \mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ and

$$L_i = L\Big|_{S_i} = C_i + \frac{g}{2} F_i \tag{2.3}$$

  where C_i and F_i are the generators of $\operatorname{Pic}(S_i)$ with $C_i^2 = -1$, $F_i^2 = 0$ and $C_i F_i = 1$ for $i = 1, 2$;
- if $g = 2$, we take $S_i \cong \mathbb{P}^2$ and $L_i = \mathcal{O}_{S_i}(1)$.

For a general choice of W, the deformation theory tells us [CLM] that W has $\lambda = K_{S_1}^2 + K_{S_2}^2$ rational double points $p_1, p_2, ..., p_\lambda \in D$ satisfying

$$\mathcal{O}_D(\sum_{j=1}^{\lambda} p_j) = N_{D/S_1} \otimes N_{D/S_2} = \mathcal{O}_D(-K_{S_1} - K_{S_2}) \tag{2.4}$$

where N_{D/S_i} are the normal bundles of D in S_i for $i = 1, 2$ and

$$\lambda = \begin{cases} 16 & \text{if } g \geq 3 \\ 18 & \text{if } g = 2. \end{cases} \tag{2.5}$$

That is, W is locally given by

$$xy = tz \tag{2.6}$$

at a point $p \in \Lambda = \{p_1, p_2, ..., p_\lambda\}$, where the surfaces S_i are locally given by $S_1 = \{x = t = 0\}$ and $S_2 = \{y = t = 0\}$, respectively.

Remark 2.1. For a general choice of W, we have a left exact sequence

$$0 \to \mathbb{Z}^{\oplus 2} \to \operatorname{Pic}(S_1) \oplus \operatorname{Pic}(S_2) \oplus \mathbb{Z}^{\oplus \lambda} \xrightarrow{\rho} \operatorname{Pic}(D) \tag{2.7}$$

where ρ is the map given by

$$\rho(M_1, M_2, n_1, n_2, ..., n_\lambda) = M_1 - M_2 + \sum n_i p_i \tag{2.8}$$

with kernel freely generated by $(L_1, L_2, 0, 0, ..., 0)$ and $(K_{S_1}, -K_{S_2}, 1, 1, ..., 1)$.

We can resolve the singularities of W by blowing up W along S_1. Let $X \to W$ be the blowup. It is not hard to see that the central fiber of X/Δ is $X_0 = R_1 \cup R_2$, where R_1 is the blowup of S_1 at Λ and $R_2 \cong S_2$. At each point $p \in \Lambda$, X is a small resolution of $p \in W$. Note that a small resolution of a 3-fold rational double point usually results in a non-Kähler complex manifold. However, in this case, X is obviously projective over Δ since W is projective over Δ and X is obtained from W by blowing up along a closed

subscheme. More explicitly, we have the line bundle $lL - R_1$ on X which is relatively ample over Δ for some large l. Here we continue to use D and L to denote the intersection $R_1 \cap R_2$ and the pullback of L from W to X, respectively.

2.2. **Generality of W.** Of course, we choose W to be very general. Actually, we can be very precise on how general W should be. We pick W such that

- D satisfies (1.5) and
- (2.7) holds, or equivalently, we have a left exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Pic}(X_0) \arrow[r] \arrow[d, equal] & \mathrm{Pic}(R_1) \oplus \mathrm{Pic}(R_2) \arrow[r, "\rho"] \arrow[d, equal] & \mathrm{Pic}(D) \\ & \mathbb{Z}^{\oplus 2} & \mathbb{Z}^{\oplus 20} & \end{tikzcd} \tag{2.9}$$

where $\rho(M_1, M_2) = M_1 - M_2$ for $M_i \in \mathrm{Pic}(R_i)$ and the kernel of ρ is freely generated by (L_1, L_2) and $(K_{R_1}, -K_{R_2})$.

These requirements on W (actually on $X_0 = R_1 \cup R_2$) are all we need to make our later argument work.

2.3. **Small resolutions of rational double points and flops.** Of course, we may also resolve the singularities of W by blowing up W along S_2 with $\widehat{X}$ the resulting 3-fold. Indeed, if we drop the requirement of projectivity, we have a choice of two small resolutions at each $p \in \Lambda$. This will result in different birational smooth models of W; they are not projective with the exception of X and $\widehat{X}$ and they can be obtained from X by a sequence of flops.

In addition, we can construct other birational "models" of W via flops; these are complex 3-folds birational to W with Picard rank 2. For example, we may start with X and let $C \subset R_1$ be a (-1)-curve on R_1. That is, C is a smooth rational curve on R_1 with $K_{R_1} \cdot C = -1$. Since R_1 is the blowup of S_1 at $\lambda \geq 9$ points, it is well known that there are infinitely many (-1)-curves on R_1. By the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & N_{C/R_1} \arrow[r] \arrow[d, equal] & N_{C/X} \arrow[r] & N_{R_1/X}\big|_C \arrow[r] \arrow[d, equal] & 0 \\ & \mathcal{O}_C(-1) & & \mathcal{O}_C(-1) & \end{tikzcd} \tag{2.10}$$

we see that $N_{C/X} \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. Hence X is locally isomorphic to an analytic neighborhood of the zero section of $N_{C/X}$ along C. Consequently, we can contract C to a rational double point, whose resolutions lead to a flop $X \dashrightarrow X'$.

We can even construct a sequence of flops along a chain of rational curves. In the more general setting, let X be a flat projective family of surfaces over Δ. Suppose that X is smooth, X_0 has simple normal crossing and there is a chain of rational curves $G_1 \cup G_2 \cup ... \cup G_n \subset X_0$ satisfying that

- $G_k \cong \mathbb{P}^1$;
- $G_k \cap G_l = \emptyset$ for $|k - l| > 1$;
- $R_k \neq R_{k+1}$, where R_k is the unique component of X_0 containing G_k;
- $G_k(X_0 - R_k) = q_{k-1} + q_k$ for $1 \leq k \leq n$, where $q_1, q_2, ..., q_n$ are n distinct points and $q_0 = \emptyset$;
- $N_{G_1/R_1} = \mathcal{O}_{G_1}(-1)$ and $N_{G_k/R_k} = \mathcal{O}_{G_k}$ for $k \geq 2$.

There is a flop $X_1 = X \dashrightarrow X_2$ about G_1. The proper transform of G_2 in X_2, which we still denote by G_2, is a (-1)-curve on the proper transform of R_2. So there is a flop $X_2 \dashrightarrow X_3$ about G_2. Continuing this process, we obtain a sequence of flops $X_1 \dashrightarrow X_2 \dashrightarrow ... \dashrightarrow X_n \dashrightarrow X_{n+1}$.

2.4. **Resolution of indeterminacy.** Suppose that there is a nontrivial dominant rational map $\phi_t : X_t \dashrightarrow X_t$ for every $t \neq 0$ given by a linear series in $|lL|$ for some fixed positive integer l. As mentioned in the previous section, we can extend it to a dominant rational map $\phi : X \dashrightarrow X$, after a base change, with the commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\phi"] \arrow[d] & X \arrow[dl] \\ \Delta & \end{tikzcd} \tag{2.11}$$

Note that after a base change, X is locally given by

$$xy = t^m \tag{2.12}$$

for some positive integer m at every point $p \in D$. So X is $\mathbb{Q}$-factorial and has canonical singularities along D. In particular, the divisors $R_i \subset X$ are $\mathbb{Q}$-Cartier (mR_i are Cartier) and $\mathrm{Pic}_{\mathbb{Q}}(X)$ is generated by L and R_i, i.e.,

$$\mathrm{Pic}_{\mathbb{Q}}(X) = \mathbb{Q}L \oplus \mathbb{Q}R_1. \tag{2.13}$$

The indeterminacy of ϕ can be resolved by a sequence of blowups along smooth centers. Let $f : Y \to X$ be the resulting birational regular map with the commutative diagram (1.3). We write $E_f = \sum E_k \subset Y$ as the union of all the exceptional divisors of f and (for convenience) the proper transforms $\widetilde{R}_i \subset Y_0$ of R_i under f for $i = 1, 2$.

Obviously, $f : Y_t \to X_t$ is a sequence of blowups at points for $t \neq 0$. We call an irreducible exceptional divisor $E \subset E_f$ a "horizontal" exceptional divisor of f if $E \not\subset Y_0$. The fiber E_t of a horizontal exceptional divisor E over every $t \neq 0 \in \Delta$ is a disjoint union of $\mathbb{P}^1$'s, which are the exceptional curves of $f : Y_t \to X_t$. So $f(E)$ is a multi-section of X/Δ. After a suitable base change, $f(E)$ becomes a union of sections of X/Δ. Consequently, E/Δ becomes a family of $\mathbb{P}^1$'s over $t \neq 0$. In addition, we can make Y_0

into a divisor of simple normal crossing after a base change by the stable reduction theorem in [KKMS]. In summary, with a suitable base change, i.e., an appropriate choice of m in (2.12), we may assume that

- Y is smooth and Y_0 has simple normal crossing;
- $E_t \cong \mathbb{P}^1$ at $t \neq 0$ for every horizontal exceptional divisor E of f;
- E_f has simple normal crossing;
- the map $Y \to W$ (via $Y \xrightarrow{f} X \to W$) also factors through $\widehat{X}$ (see 2.3) and the corresponding map $\widehat{f} : Y \to \widehat{X}$ also resolves the rational map $\widehat{\phi} : \widehat{X} \dashrightarrow \widehat{X}$; so we have the following commutative diagram:

$$\begin{tikzcd}
 & X \arrow[r, dashed, "\phi"] \arrow[d] & X \arrow[d] \\
Y \arrow[ur, "f"] \arrow[urr, bend left=60, "\varphi"] \arrow[r] \arrow[dr, "\widehat{f}"'] \arrow[drr, bend right=60, "\widehat{\varphi}"'] & W \arrow[r, dashed] & W \\
 & \widehat{X} \arrow[r, dashed, "\widehat{\phi}"] \arrow[u] & \widehat{X} \arrow[u]
\end{tikzcd} \tag{2.14}$$

The reason that we need (2.14) will be clear later.

Let $\omega_{X/\Delta}$ and $\omega_{Y/\Delta}$ be the relative dualizing sheaves of X and Y over Δ, respectively. The following identity plays a central role in our argument:

$$\omega_{Y/\Delta} = f^*\omega_{X/\Delta} + \sum \mu_k E_k = \sum \mu_k E_k = \varphi^*\omega_{X/\Delta} + \sum \mu_k E_k \tag{2.15}$$

for some integers $\mu_k = \mu(E_k) = a(E_k, X)$, which are the discrepancies of E_k with respect to X. Note that $\omega_{X/\Delta}$ is trivial. For convenience, we define $\mu(\widetilde{R}_i) = 0$.

Since X has at worse canonical singularities, we see that $\mu(E) \geq 0$ for all $E \subset E_f$. And we claim that

Proposition 2.2. *For a component $E \subset Y_0$,*

$$\mu(E) > 0 \Rightarrow \varphi_* E = 0. \tag{2.16}$$

To see this, we apply the following simple observation.

Lemma 2.3. *Let X/Δ and Y/Δ be two flat families of complex analytic varieties of the same dimension over the disk Δ. Suppose that X has reduced central fiber X_0 and Y is smooth. Let $\varphi : Y \to X$ be a proper surjective holomorphic map preserving the base. Let $S \subset Y_0$ be a reduced irreducible component of Y_0 with $\varphi_* S \neq 0$. Suppose that φ is ramified along S with ramification index $\nu > 1$. Then S has multiplicity ν in Y_0. In particular, Y_0 is nonreduced along S.*

Proof. The problem is entirely local. Let $R = \varphi(S)$, q be a general point on S and $p = \varphi(q)$. Let U be an analytic open neighborhood of p in X

and let V be the connected component of $\varphi^{-1}(U)$ that contains the point q. We may replace X and Y by U and V, respectively. Then we reduce it to the case that R and S are the only components of X_0 and Y_0, respectively, R and S are smooth and $\varphi : S \to R$ is an isomorphism, in which case the lemma follows easily. □

Proof of Proposition 2.2. If $\varphi_* E \neq 0$, then φ is ramified along E with ramification index $\mu(E) + 1$ by (2.15) and Riemann-Hurwitz. This is impossible unless $\mu(E) = 0$ by the above lemma and (2.16) follows. □

We let $\mathcal{S} \subset Y_0$ be the union of components E with $\mu(E) = 0$, i.e.,

$$\mathcal{S} = \sum_{\mu_k = 0} E_k. \tag{2.17}$$

Then it follows from (2.16) that

$$\varphi_* \mathcal{S} = (\deg \phi)(R_1 + R_2). \tag{2.18}$$

Since X is smooth outside of D, $\mu(E) > 0$ if $f(E) \not\subset D$ and $f_* E = 0$. Consequently, we have $f(E) \subset D$ for every component $E \subset \mathcal{S}$ with $f_* E = 0$. Note that $\widetilde{R}_i \subset \mathcal{S}$.

Actually, we can arrive at a more precise picture of $\mathcal{S}$ as follows.

2.5. **Structure of $\mathcal{S}$.** We may resolve the singularities of X by repeatedly blowing up X along R_1. By that we mean we first blow up X along R_1, then we blow up the proper transform of R_1 and so on. Let $\eta : X' \to X$ be the resulting resolution. We see that

$$X_0' = P_0 \cup P_1 \cup ... \cup P_{m-1} \cup P_m \tag{2.19}$$

where P_0 and P_m are the proper transforms of R_1 and R_2, respectively, P_i are ruled surfaces over D for $0 < i < m$ and $P_i \cap P_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Note that the relative dualizing sheaf of X'/Δ satisfies

$$\omega_{X'/\Delta} = \eta^* \omega_{X/\Delta} \tag{2.20}$$

and hence remains trivial.

We claim that $f : Y \to X$ factors through X'. Again, the problem is local. It is enough to prove the following.

Lemma 2.4. *Let X be the n-fold singularity given by $x_1 x_2 = t^m$ $(m \geq 1)$ in $\Delta^{n+1}_{x_1 x_2 \ldots x_n t}$ and let $\eta : X' \to X$ be the desingularization of X obtained by repeatedly blowing up X along $R_1 = \{x_1 = t = 0\}$. Let $f : Y \to X$ be another desingularization of X with $Y_0 = f^*(X_0)$ supported on a divisor of normal crossing. Here a desingularization $f : Y \to X$ of X is a proper birational map from a smooth variety Y to X. Then $f : Y \to X$ factors through X'.*

Proof. Let $R_2 = \{x_2 = t = 0\}$, $D = R_1 \cap R_2$ and p be the origin.

Basically, we want to show that the rational map

$$f' = \eta^{-1} \circ f : Y \dashrightarrow X' \tag{2.21}$$

is regular. Let

$$X' = X_m \to X_{m-1} \to ... \to X_2 \to X_1 = X \tag{2.22}$$

be the sequence of blowups over R_1, where $X_k \to X_{k-1}$ is the blowup along the proper transform of R_1. It is easy to see that X_k has singularities of type $x_1x_2 = t^{m-k+1}$. Hence we may proceed by induction on m and it suffices to prove that the rational map $Y \dashrightarrow X_2$ is regular. So we may replace X' by X_2. It is easy to see that $X_0' = \widetilde{R}_1 \cup P \cup \widetilde{R}_2$, where $\widetilde{R}_i$ are the proper transforms of R_i and $P \cong D \times \mathbb{P}^1$.

We can resolve the indeterminacy of f' by a sequence of blowups with smooth centers. That is, we have

$$Z = Y_{l+1} \xrightarrow{\nu_l} Y_l \xrightarrow{\nu_{l-1}} ... \xrightarrow{\nu_2} Y_2 \xrightarrow{\nu_1} Y_1 = Y \tag{2.23}$$

where $\nu_k : Y_{k+1} \to Y_k$ is the blowup of Y_k centered at a smooth irreducible subvariety $F_k \subset Y_k$. The resulting map $\varepsilon : Z \to X'$ is regular. Namely, we have the commutative diagram:

$$\begin{tikzcd} & Z \arrow[dl, "\varepsilon"'] \arrow[dr, "\nu"] & \\ X' \arrow[dr, "\eta"'] & & Y \arrow[ll, dashed, "f'"'] \arrow[dl, "f"] \\ & X & \end{tikzcd} \tag{2.24}$$

where $\nu = \nu_1 \circ \nu_2 \circ ... \circ \nu_l$.

In addition, we may choose the sequence of blowups that all Y_k have simple normal crossing supports on the central fiber.

Let $f_k = f \circ \nu_1 \circ \nu_2 \circ ... \circ \nu_{k-1}$ and $f_k' = f' \circ \nu_1 \circ \nu_2 \circ ... \circ \nu_{k-1}$ for $k = 1, 2, ..., l$.

Let $E_k = \nu_k^{-1}(F_k) \subset Y_{k+1}$ be the exceptional divisor of ν_k for $k = 1, 2, ..., l$. We will show inductively that the map $Y_{k+1} \to X'$ contracts the fibers of E_k/F_k.

If F_l has codimension > 2 in Y_l, then E_l is a $\mathbb{P}^e$ bundle over F_l with $e \geq 2$. Consider the image of E_l under the regular map $\varepsilon : Z \to X'$. A proper map $\mathbb{P}^e \to X_0'$ must be constant if $e \geq 2$. Therefore, ε contracts E_l along the fibers of E_l/F_l.

Suppose that F_l has codimension 2 in Y_l. Then E_l is a $\mathbb{P}^1$ bundle over F_l. Suppose that ε does not contract the fibers of E_l/F_l. Obviously, $\varepsilon(E_l) \subset P$ and ε maps every fiber of E_l/F_l onto a fiber of P/D. Hence f_l' is not regular along F_l.

Clearly, f_l' is regular at every point $q \notin f_l^{-1}(D)$. So $f_l(F_l) \subset D$. Let $q \in F_l$ be a general point of F_l. WLOG, we may simply assume that $p = f_l(q)$. Then the map $f_l : Y_l \to X$ induces a map on the local rings of (analytic) functions

$$f_l^{\#} : \mathcal{O}_p \cong \mathbb{C}[[x_1, x_2, ...x_n, t]]/(x_1x_2 - t^m) \to \mathcal{O}_q. \tag{2.25}$$

Since f_l is birational, $f_l^{\#}$ induces an isomorphism on the function fields, i.e.,

$$f_l^{\#} : K(\mathcal{O}_p) \cong \mathbb{C}((x_1, x_3, ..., x_n, t)) \xrightarrow{\sim} K(\mathcal{O}_q). \tag{2.26}$$

Since f_l' is not regular at q, we necessarily have

$$\frac{f_l^{\#}(x_1)}{f_l^{\#}(t)} \notin \mathcal{O}_q \text{ and } \frac{f_l^{\#}(t)}{f_l^{\#}(x_1)} \notin \mathcal{O}_q. \tag{2.27}$$

Since F_l has codimension two in Y_l, it is either a component of the intersection of two distinct components of $(Y_l)_0 = Y_l \cap \{t = 0\}$ or not contained in the intersection of two distinct components of $(Y_l)_0$.

Suppose that F_l is not contained in the intersection of two distinct components of $(Y_l)_0$. Then Y_l is locally given by $u^\alpha = t$ at q for some positive integer α. It is easy to see that $f_l^{\#}(x_1) = u^a$ and $f_l^{\#}(t) = u^\alpha$ for some positive integers $a < m\alpha$. Clearly, (2.27) cannot hold. Contradiction.

Suppose that F_l is a component of the intersection of two distinct components of $(Y_l)_0$. Then Y_l is locally given by $u^\alpha v^\beta = t$ at q for some positive integers α and β with $\mathcal{O}_q \cong \mathbb{C}[[u, v, w_1, w_2, ..., w_{n-2}]]$. Hence

$$f_l^{\#}(x_1) = u^a v^b \text{ and } f_l^{\#}(t) = u^\alpha v^\beta \tag{2.28}$$

for some integers $0 \le a \le m\alpha$ and $0 \le b \le m\beta$. Obviously, (2.27) holds if $a < \alpha$ and $b > \beta$ or $a > \alpha$ and $b < \beta$. WLOG, suppose that $a < \alpha$ and $b > \beta$. We observe that $\alpha b - a\beta \ge 2$. Then it is not hard to see

$$\begin{aligned} f_l^{\#}(K(\mathcal{O}_p)) &= \mathbb{C}((u^a v^b, u^\alpha v^\beta, f_l^{\#}(x_3), ..., f_l^{\#}(x_n))) \\ &\subsetneq \mathbb{C}((u, v, w_1, w_2, ..., w_{n-2})) = K(\mathcal{O}_q). \end{aligned} \tag{2.29}$$

Contradiction.

In conclusion, ε contracts the fibers of E_l/F_l and hence $f_l' : Y_l \dashrightarrow X'$ is regular. Repeating this argument, we conclude that $f_1' = f' : Y = Y_1 \dashrightarrow X'$ is regular. $\square$

Therefore, we have the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[d, "\varepsilon"'] \arrow[dr, "f"] & X \\ X' \arrow[r, "\eta"] & X \arrow[u, dashed, "\phi"'] \end{tikzcd} \tag{2.30}$$

Remark 2.5. We can do the same for $\widehat{f} : Y \to \widehat{X}$. Let $\widehat{X}'$ be the resolution of singularities of $\widehat{X}$ by repeatedly blowing up along $\widehat{R}_1$, where $\widehat{X}_0 = \widehat{R}_1 \cup \widehat{R}_2$

with $\widehat{R}_i$ the proper transform of R_i. Then we have the commutative diagram

$$\begin{tikzcd}
Y \arrow[r, equal] \arrow[d, "\widehat{\varepsilon}"'] \arrow[dd, bend right=60, "\widehat{f}"'] & Y \arrow[d, "\varepsilon"] \arrow[dd, bend left=60, "f"] \\
\widehat{X}' \arrow[r, dashed] \arrow[d, "\widehat{\eta}"'] & X' \arrow[d, "\eta"] \\
\widehat{X} \arrow[r, dashed, "\xi"] & X
\end{tikzcd} \tag{2.31}$$

which can be put together with (2.14) into

$$\begin{tikzcd}
X' \arrow[r, "\eta"] & X \arrow[r, dashed, "\phi"] \arrow[d] & X \arrow[d] \\
Y \arrow[u, "\varepsilon"] \arrow[d, "\widehat{\varepsilon}"'] \arrow[r] \arrow[ur, "f"] \arrow[urr, "\varphi"'] \arrow[dr, "\widehat{f}"'] \arrow[drr, "\widehat{\varphi}"] & W \arrow[r, dashed] & W \\
\widehat{X}' \arrow[r, "\widehat{\eta}"'] & \widehat{X} \arrow[r, dashed, "\widehat{\phi}"'] \arrow[u] & \widehat{X} \arrow[u]
\end{tikzcd} \tag{2.32}$$

Let $I_p \subset P_0 \cong R_1$ be the exceptional curve of $R_1 \to S_1$ over a point $p \in \Lambda$ and let $I_{i,p}$ be the fiber of $\eta : P_i \to D$ over a point $p \in D$ for $0 < i < m$. Then $\widehat{X}'$ can be alternatively constructed as the manifold obtained from X by the sequences of flops along

$$\bigcup_{p\in\Lambda} \bigcup_{i=0}^{m-1} I_{i,p} \tag{2.33}$$

where we let $I_{0,p} = I_p$.

Let $Q_i \subset Y$ be the proper transforms of P_i under ε for $i = 0, 1, ..., m$. Note that $Q_0 = \widetilde{R}_1$ and $Q_m = \widetilde{R}_2$. Since X' is smooth, every exceptional divisor of ε has discrepancy at least 1. Therefore, Q_i are the only components of Y_0 with $\mu(Q_i) = 0$. Consequently,

$$\mathcal{S} = Q_0 + Q_1 + ... + Q_{m-1} + Q_m, \tag{2.34}$$

$$f(Q_i) = D \text{ for } 0 < i < m \tag{2.35}$$

and

$$\varphi_* \mathcal{S} = \sum_{i=0}^{m} \varphi_* Q_i = (\deg \phi)(R_1 + R_2). \tag{2.36}$$

Obviously, Q_i is birational to $D \times \mathbb{P}^1$ for each $0 < i < m$.

Let S be a component of Y_0. Then by (2.15) and adjunction, we have

$$\omega_S = (\omega_{Y/\Delta} + S)\Big|_S = \sum \mu_k E_k\Big|_S - \sum_{\substack{E_k \neq S \\ E_k \subset Y_0}} E_k\Big|_S \tag{2.37}$$

and hence

$$\sum_{E_k \not\subset Y_0} \mu_k E_k \Big|_S = \omega_S + \sum_{\substack{E_k \neq S \\ E_k \subset Y_0}} (1 + \mu(S) - \mu_k) E_k \Big|_S \tag{2.38}$$

Suppose that $S = Q \subset \mathcal{S}$. Then (2.37) becomes

$$\sum_{\substack{E_k \neq Q \\ E_k \subset Y_0}} (1 - \mu_k) E_k \Big|_Q = -\omega_Q + \sum_{E_k \not\subset Y_0} \mu_k E_k \Big|_Q \tag{2.39}$$

Suppose that $Q \neq Q_0, Q_m$. Let $Q_p \cong \mathbb{P}^1$ be the fiber of $f : Q \to D$ over a general point $p \in D$. Clearly, we have

$$Q_p \cdot \omega_Q = -2 \tag{2.40}$$

and hence

$$\sum_{\substack{E_k \neq Q \\ E_k \subset Y_0}} (1 - \mu_k) E_k \cdot Q_p \geq 2 \tag{2.41}$$

Therefore, each Q_j $(0 < j < m)$ meets at least two other Q_i $(0 \leq i \leq m)$ along sections of Q_j/D; and since Q_j is the proper transform of P_j, it cannot meet more than two among Q_i. So we see that Q_i form a "chain" in the same way as P_i do. More precisely, we have

- Q_i and Q_{i+1} meet transversely along a curve $D_i = Q_i \cap Q_{i+1} \cong D$ for $0 \leq i < m$;
- $D_i \cong D$ are sections of Q_i/D for $1 \leq i \leq m-1$ and Q_{i+1}/D for $0 \leq i \leq m-2$.
- $Q_i \cap Q_j = \emptyset$ for $|i - j| > 1$.

Next, we claim that

Proposition 2.6. *For each $0 \leq i \leq m$, we have*

$$\textit{either } \varphi_* Q_i \neq 0 \textit{ or } \varphi(Q_i) = D. \tag{2.42}$$

Namely, every Q_i either dominates one of R_1 and R_2 or is contracted onto D by φ. Since $\widetilde{R}_i$ cannot be mapped onto D, this implies that

$$\varphi_* \widetilde{R}_i \neq 0 \tag{2.43}$$

for $i = 1, 2$. So φ does not contract $\widetilde{R}_i$, as pointed out in the very beginning of the paper.

Note that if X were smooth, we would already have that $\varphi_* S \neq 0$ for all S with $\mu(S) = 0$ by (2.15) and Riemann-Hurwitz. However, things are a little more subtle here since X is singular.

Proof of Proposition 2.6. A natural thing to do is to resolve the indeterminacy of the rational map $\phi' = \eta^{-1} \circ \phi \circ \eta : X' \dashrightarrow X'$ with the diagram

$$\begin{tikzcd} Y' \arrow[d, "\varepsilon'"'] \arrow[dr, "\varphi'"] & \\ X' \arrow[r, dashed, "\phi'"] \arrow[d, "\eta"'] & X' \arrow[d, "\eta"] \\ X \arrow[r, dashed, "\phi"] & X \end{tikzcd} \tag{2.44}$$

where we can make Y_0' have simple normal crossing support. Let $Q_i' \subset Y'$ be the proper transforms of P_i under ε'. Obviously, Q_i' are the proper transforms of Q_i under the rational map $\varepsilon^{-1} \circ \varepsilon' : Y' \dashrightarrow Y$. To show that (2.42) holds for Q_i, it suffices to show that the same thing holds for Q_i' when we map Y' to X via $\eta \circ \varphi'$.

Let μ' be the discrepancy function corresponding to the map $\eta \circ \varepsilon'$, i.e.,

$$\omega_{Y'/\Delta} = (\varepsilon')^* \eta^* \omega_{X/\Delta} + \sum \mu'(E) E \tag{2.45}$$

where E runs through all exceptional divisors of $\eta \circ \varepsilon'$ and all components of Y_0'. By (2.20), we see that $\mu'(Q_i') = 0$ for all $0 \le i \le m$. Then we have $\varphi'_* Q_i' \neq 0$ by Riemann-Hurwitz and the fact that X' is smooth. So each Q_i' dominates some P_j via φ'. If Q_i' dominates P_0 or P_m, then Q_i' dominates R_1 or R_2 via $\eta \circ \varphi'$; if Q_i' dominates P_j for some $0 < j < m$, then $\eta(\varphi'(Q_i')) = D$. This proves (2.42). □

2.6. **The map $\mathcal{S} \to X_0$.** Let us consider the restriction of φ to $\mathcal{S}$, i.e., $\varphi_{\mathcal{S}} : \mathcal{S} \to X_0$. Again by (2.15) and adjunction,

$$\begin{aligned} \omega_{\mathcal{S}} &= (\omega_{Y/\Delta} + \mathcal{S})\Big|_{\mathcal{S}} = (\omega_{Y/\Delta} - \mathcal{T})\Big|_{\mathcal{S}} \\ &= \sum_{E_k \not\subset Y_0} \mu_k E_k\Big|_{\mathcal{S}} + \sum_{E_k \subset \mathcal{T}} (\mu_k - 1) E_k\Big|_{\mathcal{S}} \\ &= \varphi_{\mathcal{S}}^* \omega_{X_0} + \sum_{E_k \not\subset Y_0} \mu_k E_k\Big|_{\mathcal{S}} + \sum_{E_k \subset \mathcal{T}} (\mu_k - 1) E_k\Big|_{\mathcal{S}} \end{aligned} \tag{2.46}$$

where

$$\mathcal{T} = Y_0 - \mathcal{S} = \sum_{\substack{\mu_k > 0 \\ E_k \subset Y_0}} E_k. \tag{2.47}$$

This gives us the discriminant locus of $\varphi_{\mathcal{S}}$. It is also easy to see the following from (2.46).

Proposition 2.7. *If i and j are two integers satisfying that $0 \le i < j \le m$, $\varphi_* Q_i \neq 0$, $\varphi_* Q_j \neq 0$ and $\varphi_* Q_\alpha = 0$ for all $i < \alpha < j$, then we have either $\varphi(Q_i) = R_1$ and $\varphi(Q_j) = R_2$ or $\varphi(Q_i) = R_2$ and $\varphi(Q_j) = R_1$; in other*

words, Q_i and Q_j cannot dominate the same R_n via φ for $n = 1, 2$. As a consequence, $\varphi(D_i) = D$ for all $0 \le i < m$.

Proof. We leave the proof to the readers. $\square$

2.7. **Invariants α_i and β_i.** Let $Q = Q_i$ be a component of $\mathcal{S}$. Suppose that Q dominates $R = R_j$ for some $1 \le j \le 2$. Let $\varphi_{Q_i} = \varphi_Q : Q \to R$ be the restriction of φ to Q. We may put (2.39) in the form of log version Riemann-Hurwitz

$$\begin{aligned}
&\omega_Q + D_{i-1} + D_i \\
&= \varepsilon_Q^*(\omega_P + \varepsilon_* D_{i-1} + \varepsilon_* D_i) + \sum_{E_k \subset \mathcal{T}} (\mu_k - 1) E_k \bigg|_Q + \sum_{E_k \not\subset Y_0} \mu_k E_k \bigg|_Q \\
&= \varphi_Q^*(\omega_R + D) + \sum_{E_k \subset \mathcal{T}} (\mu_k - 1) E_k \bigg|_Q + \sum_{E_k \not\subset Y_0} \mu_k E_k \bigg|_Q
\end{aligned} \tag{2.48}$$

where we set $D_{-1} = D_m = \emptyset$ and $\varepsilon_{Q_i} = \varepsilon_Q : Q \to P_i = P$ is the restriction of ε to Q. We let α_i and β_{i-1} be the multiplicities of D_i and D_{i-1} in $\varphi_{Q_i}^* D$, respectively. Here we set $\alpha_i = \beta_{i-1} = 0$ if $\varphi_* Q_i = 0$. Obviously, α_i and β_{i-1} are the ramification indices of φ_{Q_i} along D_i and D_{i-1}, respectively. We claim that

$$\deg \varphi_{Q_i} = \alpha_i \deg \varphi_{D_i} + \beta_{i-1} \deg \varphi_{D_{i-1}} \tag{2.49}$$

where $\varphi_{D_i} : D_i \to D$ is the restriction of φ to D_i and we set $\alpha_m = \beta_{-1} = 0$. This is a consequence of the following observation.

Proposition 2.8. *Let $Q = Q_i \subset \mathcal{S}$ be a component satisfying $\varphi_* Q \neq 0$. Then $\varphi_Q^{-1}(D) = D_{i-1} \cup D_i \cup \Gamma$ with $\varphi_* \Gamma = 0$. This still holds if we replace φ by $\widehat{\varphi}$.*

Proof. Otherwise, there is an irreducible curve $\Gamma \subset Q$ such that $\Gamma \neq D_{i-1}$, $\Gamma \neq D_i$ and $\varphi(\Gamma) = D$. So Γ is not rational and hence $\varepsilon_* \Gamma \neq 0$. Let Σ be the union of components of Y_0 that dominate $G = \varepsilon(\Gamma)$ via ε. Let q be a general point on Γ, $p = \varepsilon(q)$ and $J = \varepsilon^{-1}(p)$. By Zariski's main theorem, J is connected and hence Σ is connected. Let $J_1 \subset J$ be the component of J containing q and Let $\Sigma_1 \subset \Sigma$ be the component of Y_0 containing J_1. Obviously, $\Gamma \subset \Sigma_1$ and hence $D \subset \varphi(\Sigma_1)$. And since $\Sigma_1 \subset \mathcal{T}$, $\varphi_* \Sigma_1 = 0$. Therefore, $\varphi(\Sigma_1) = D$. And since $J_1 \cong \mathbb{P}^1 \subset \Sigma_1$, $\varphi_* J_1 = 0$ and φ contracts Σ_1 onto D along the fibers of $\varepsilon : \Sigma_1 \to G$. Let $J_2 \neq J_1 \subset J$ be a component of J with $J_1 \cap J_2 \neq \emptyset$ and let $\Sigma_2 \subset \Sigma$ be the component of Y_0 containing J_2. Then Σ_2 meets Σ_1 along a multi-section of Σ_1/G. Therefore, $\varphi(\Sigma_2) = D$ and $\varphi_* J_2 = 0$ by the same argument as before. We can argue this way inductively that $\varphi_* J = 0$ and $\varphi(\Sigma^\circ) = D$ for every component $\Sigma^\circ \subset \Sigma$. Let $r = \varphi(q)$ and K be the connected component of $\varphi^{-1}(r)$ containing the point q. Obviously, $J \subset K$. We claim that $J = K$. Otherwise, there is a component $K^\circ \subset K$ such that $K^\circ \not\subset J$ and $K^\circ \cap J \neq \emptyset$. Let T be a component of Y_0 containing K°. Obviously, $T \not\subset \Sigma$; otherwise, we

necessarily have $\varepsilon(K^\circ) = p$ and $K^\circ \subset J$. Also we cannot have $T = Q$; otherwise, $K^\circ \subset Q$, $q \in K^\circ$ and $\varphi_* K^\circ = 0$, which is impossible for a general point $q \in \Gamma$. We cannot have $T = Q'$ for some $Q' \neq Q \subset \mathcal{S}$, either, since $p \in \varepsilon(K^\circ) \subset \varepsilon(T)$. Therefore, $T \subset \mathcal{T}$. Since $K^\circ \cap J \neq \emptyset$, $T \cap \Sigma \neq \emptyset$. If T and Σ meet along a multi-section of Σ/G, $\varepsilon(T) = G$, which is impossible as we have proved that $T \not\subset \Sigma$. Therefore, $T \cap \Sigma$ is contained in the fibers of Σ/G. And since $T \cap J \neq \emptyset$, $T \cap \Sigma$ contains a component of J, which is impossible for a general point $p \in G$. Therefore, $J = K$.

Let $U \subset X$ be an analytic open neighborhood of r in X and $V \subset Y$ be the connected component of $\varphi^{-1}(U)$ containing J. Since $\varepsilon(J) = p \notin P_{i-1} \cup P_{i+1}$, $V \cap \mathcal{S} = V \cap Q$. Therefore, $\varphi_* M = 0$ for all components M of V_0 with $M \neq V \cap Q$. So V_0 cannot dominate U_0. Contradiction. $\square$

Actually α_i and β_{i-1} are determined as follows.

Proposition 2.9. *Let $0 \le i < j \le m$ be two integers with the properties that $\varphi_* Q_i \neq 0$, $\varphi_* Q_j \neq 0$ and $\varphi_* Q_k = 0$ for all $i < k < j$. Then*

$$\alpha_i = \beta_{j-1} = \frac{m}{j-i}. \tag{2.50}$$

Proof. Let q be a general point on D_i and $U \subset X$ be an analytic open neighborhood of $\varphi(q)$ in X. Let $V \subset Y$ be the connected component of $\varphi^{-1}(U)$ containing q. Let $T \subset V_0$ be a component of $V \cap \mathcal{T}$. Since q is a general point on D_i, it is easy to see that $\varepsilon(T) \cap (P_i \cup P_j) = \emptyset$ and $\varepsilon(T) \subset P_k$ for some $i < k < j$. Indeed, $\varepsilon(T)$ is a multi-section of P_k/D, T is a $\mathbb{P}^1$ bundle over $\varepsilon(T)$ and φ contracts the fibers of $T/\varepsilon(T)$ and maps T onto $D \cap U$. Therefore, $\varepsilon : V \to V' = \varepsilon(V)$ is proper and U' is open in X'. In addition, since T is contracted by φ along the fibers of $T/\varepsilon(T)$, the rational map $\phi = \varphi \circ \varepsilon^{-1} : V' \dashrightarrow U$ is actually regular. Furthermore, ϕ contracts P_k for all $i < k < j$ and hence we have the diagram

$$\begin{tikzcd} V' \arrow[r, "\phi"] \arrow[d, "\tau"'] & U \\ V'' \arrow[ur] & \end{tikzcd} \tag{2.51}$$

where $\tau : V' \to V''$ is the birational map contracting all P_k for $i < k < j$. That is, V'' is the threefold given by $xy = t^{j-i}$ in Δ^4_{xyzt}. The map $\phi \circ \tau^{-1}$ is regular and finite and sends $V'' = \{xy = t^{j-i}\}$ onto $U = \{xy = t^m\}$ while preserving the base $\Delta = \{|t| < 1\}$. It has to be the map sending (x, y, z, t) to (x^a, y^a, z, t) with $a = m/(j-i)$. It follows that $\alpha_i = \beta_{j-1} = a$. $\square$

Corollary 2.10. *The following holds:*

- *$\alpha_i \neq 1$ and $\beta_{i-1} \neq 1$ for all $0 < i < m$.*
- *If $\deg \varphi_{Q_0} = 1$ or $\deg \varphi_{Q_m} = 1$, then $\deg \varphi = 1$.*

Proof. The first statement follows directly from Proposition 2.9.

If $\deg \varphi_{Q_0} = 1$, then $\alpha_0 = \deg \varphi_{D_0} = 1$. By Proposition 2.9, we must have $\beta_{m-1} = 1$ and $\varphi_* Q_k = 0$ for all $0 < k < m$. Hence $\deg \varphi_{D_m} = \deg \varphi_{D_0} = 1$ and $\deg \varphi_{Q_m} = 1$ by (2.49). It follows that $\deg \varphi = 1$.

Similarly, we can show that $\deg \varphi = 1$ if $\deg \varphi_{Q_m} = 1$. □

Corollary 2.11. *The following are equivalent:*

- $\alpha_0 = 1$.
- $\beta_{m-1} = 1$.
- $\alpha_i = 1$ *for some* $0 \le i \le m-1$.
- $\beta_j = 1$ *for some* $0 \le j \le m-1$.
- $\varphi_* Q_i = 0$ *for all* $1 \le i \le m-1$.
- $\deg \phi_{R_1} = \deg \phi$, *where* ϕ_{R_1} *is the restriction of* ϕ *to* R_1.
- $\deg \phi_{R_2} = \deg \phi$, *where* ϕ_{R_2} *is the restriction of* ϕ *to* R_2.

Proof. This is more or less trivial. □

3. Some local results

Before proceeding any further, we will first prove a few lemmas of local nature. Impatient readers can skip this section and only refer back when they are needed.

The first is basically Lemma 2.2 in [Ch].

Lemma 3.1. *Let X be the n-fold defined by $x_1 x_2 = t^m$ in $\Delta^{n+1}_{x_1 x_2 \ldots x_n t}$ for some integer $m > 0$, $\mathcal{C}$ be a flat family of curves over the disk $\Delta = \{|t| < 1\}$ and $\varepsilon : \mathcal{C} \to X$ be a proper map preserving the base Δ. If $\varepsilon_* \mathcal{C} \ne 0$, then there is a component $\Gamma_i \subset \mathcal{C}_0$ for each $i = 1, 2$ such that $\varepsilon_* \Gamma_i \ne 0$ and $\varepsilon(\Gamma_i) \subset R_i = \{x_i = t = 0\}$.*

Proof. Since $\varepsilon_* \mathcal{C} \ne 0$, $\dim \varepsilon(\mathcal{C}) = 2$. Note that X is $\mathbb{Q}$-factorial and R_i is a $\mathbb{Q}$-Cartier divisor on X. So $\dim(R_i \cap \varepsilon(\mathcal{C})) = 1$. It follows that there is a component $\Gamma_i \subset \mathcal{C}_0$ such that $\varepsilon_* \Gamma_i \ne 0$ and $\varepsilon(\Gamma) \subset R_i$. □

In the above lemma, when $m = 1$ and $\mathcal{C}_0$ is nodal, we can say much more.

Lemma 3.2. *Let X be the n-fold defined by $x_1 x_2 = t$ in $\Delta^{n+1}_{x_1 x_2 \ldots x_n t}$, $\mathcal{C}$ be a flat family of curves over the disk $\Delta = \{|t| < 1\}$ and $\varepsilon : \mathcal{C} \to X$ be a proper map preserving the base Δ. Suppose that $\mathcal{C}$ is smooth, $\mathcal{C}_0$ has normal crossing and $\varepsilon_* \mathcal{C} \ne 0$. Then*

(1) *$\varepsilon(\Gamma) \not\subset D = \{x_1 = x_2 = 0\}$ for every component $\Gamma \subset \mathcal{C}_0$;*
(2) *$\varepsilon_* \Gamma \ne 0$ for every component $\Gamma \subset \mathcal{C}_0$ and $\varepsilon(\Gamma) \cap D \ne \emptyset$;*
(3) *Γ meets $\varepsilon^* R_2$ transversely for every component $\Gamma \subset \mathcal{C}_0$ satisfying that $\varepsilon(\Gamma) \subset R_1$, where $R_1 = \{x_1 = 0\}$ and $R_2 = \{x_2 = 0\}$;*
(4) *if there is a component $\Gamma_1 \subset \mathcal{C}_0$ and a point $q \in \Gamma_1$ satisfying that $\varepsilon(\Gamma_1) \subset R_1$ and $p = \varepsilon(q) \in D$, then there is a component $\Gamma_2 \subset \mathcal{C}_0$ satisfying that $\varepsilon(\Gamma_2) \subset R_2$ and $q \in \Gamma_1 \cap \Gamma_2$.*

Proof. Suppose that there is a component $\Gamma \subset \mathcal{C}_0$ such that $\varepsilon(\Gamma) \subset D$. Since $\mathcal{C}_0$ is reduced along Γ, there is a section B of $\mathcal{C}/\Delta$ such that $B \cap \Gamma \ne \emptyset$.

Since ε preserves the base, $\varepsilon(B)$ is a section of X/Δ that meets D. This is impossible as X is smooth. This proves both (1) and (2).

Let $p = \{x_1 = x_2 = ... = x_n = t = 0\}$ be the origin. WLOG, we may assume that $\varepsilon : \mathcal{C} \to X$ has connected fibers, $p \in \varepsilon(\Gamma)$ for every component $\Gamma \subset \mathcal{C}_0$ and p is the only point in the intersection $\varepsilon(\mathcal{C}) \cap D$.

By (2), $\varepsilon^{-1}(p)$ is a finite set of points. So $\varepsilon^{-1}(p) = \{q\}$ consists of a single point $q \in \mathcal{C}_0$. Consequently, $\mathcal{C}_0$ has at most two components. Obviously, $\varepsilon^* R_i \neq 0$ since $\varepsilon^{-1}(R_i) \cap \mathcal{C}_0 \neq \emptyset$. Therefore, $\mathcal{C}_0 = \Gamma_1 \cup \Gamma_2$ consists of two components Γ_i with $\varepsilon(\Gamma_i) \subset R_i$. And since ε preserves the base, $\varepsilon^* R_i$ is reduced and hence $\varepsilon^* R_i = \Gamma_i$. Then it is clear that

$$\Gamma_2 \cdot \varepsilon^* R_1 = \Gamma_1 \cdot \varepsilon^* R_2 = \Gamma_1 \cdot \Gamma_2 = 1 \tag{3.1}$$

and (3) and (4) follow. □

Lemma 3.3. *Let X and Y be two flat families of analytic varieties of dimension $n-1$ over $\Delta = \{|t| < 1\}$, where $X \cong \{x_1 x_2 = t^m\} \subset \Delta^{n+1}_{x_1 x_2 \ldots x_n t}$ for some integer $m \geq 1$ and Y is smooth with simple normal crossing central fiber. Let $\mathcal{C} \subset Y$ be a flat family of curves over Δ cut out by the general members of $n-2$ base point free linear systems on Y and let $\varepsilon : Y \to X$ be a proper birational map preserving the base. Then Γ meets $\varepsilon^* R_2$ transversely for every component $\Gamma \subset \mathcal{C}_0$ satisfying that $\varepsilon(\Gamma) \subset R_1$ and $\varepsilon(\Gamma) \not\subset D$, where $R_1 = \{x_1 = 0\}$, $R_2 = \{x_2 = 0\}$ and $D = \{x_1 = x_2 = t = 0\}$.*

Proof. By Lemma 2.4, ε factors through X', where X' is the desingularization of X by repeatedly blowing up along $\{x_1 = t = 0\}$. Applying Lemma 3.2 to $\varepsilon : \mathcal{C} \to X'$, we are done. □

4. The pullback $\varphi^*\mathcal{L}$

Let $\mathcal{C} = \varphi^* \mathcal{L} \subset Y$ be the pullback of a general member

$$\mathcal{L} \in |\mathcal{L}_{\sigma_0,\sigma_1}| = |\sigma_0 L - \sigma_1 R_1| \tag{4.1}$$

on X, where σ_i are positive integers such that $\mathcal{L}_{\sigma_0,\sigma_1} = \sigma_0 L - \sigma_1 R_1$ is Cartier and very ample. Note that $\mathcal{L}_{\sigma_0,\sigma_1}$ is Cartier if and only if $m | \sigma_1$ and it can be made sufficiently ample if we choose $\sigma_0 >> \sigma_1 > 0$.

Since $\varphi^* \mathcal{L}$ is big and base point free and Y_0 has simple normal crossing, $\mathcal{C}$ is smooth and the central fiber $\mathcal{C}_0$ of $\mathcal{C}/\Delta$ is a connected curve of simple normal crossing. First, we prove the following.

Proposition 4.1. *For all $t \in \Delta$,*

$$p_a(\mathcal{C}_t) = p_a(\mathcal{C} \cap \mathcal{S}) \tag{4.2}$$

where $p_a(C)$ is the arithmetic genus of a curve C.

Proof. Clearly, since $\mathcal{C} \cap \mathcal{S} \subset \mathcal{C}_0$ and $\mathcal{C}_0$ is connected and reduced,

$$p_a(\mathcal{C}_t) = p_a(\mathcal{C}_0) \geq p_a(\mathcal{C} \cap \mathcal{S}). \tag{4.3}$$

On the other hand,

$$2p_a(\mathcal{C}_t) - 2 = (\omega_{Y/\Delta} + \mathcal{C})\mathcal{C} Y_t = \omega_{Y/\Delta} \mathcal{C} Y_t + \mathcal{C}^2 Y_t \tag{4.4}$$

and

$$2p_a(\mathcal{C}\cap\mathcal{S}) - 2 = (\omega_{\mathcal{S}} + \mathcal{C})\mathcal{C}\Big|_{\mathcal{S}} = \omega_{\mathcal{S}}\mathcal{C}\Big|_{\mathcal{S}} + \mathcal{C}^2\mathcal{S}. \tag{4.5}$$

Note that

$$\mathcal{C}^2 Y_t = \mathcal{C}^2 Y_0 = \mathcal{C}^2(\mathcal{S}+\mathcal{T}) = \mathcal{C}^2\mathcal{S} \tag{4.6}$$

since $\varphi_*\mathcal{T} = 0$. So it suffices to show that

$$\omega_{\mathcal{S}}\mathcal{C}\Big|_{\mathcal{S}} \ge \omega_{Y/\Delta}\mathcal{C}Y_t = \sum_{E_k\not\subset Y_0} \mu_k E_k \mathcal{C}_0 = \sum_{E_k\not\subset Y_0} \mu_k E_k \mathcal{C}(\mathcal{S}+\mathcal{T}) \tag{4.7}$$

by (2.15). Combining (4.7) and (2.46), it comes down to prove that

$$\sum_{E_k\subset\mathcal{T}} (\mu_k - 1)E_k\mathcal{S}\mathcal{C} \ge \sum_{E_k\not\subset Y_0} \mu_k E_k\mathcal{T}\mathcal{C}. \tag{4.8}$$

By (2.46), the dualizing sheaf $\omega_{\mathcal{C}\cap\mathcal{S}}$ is given by

$$\begin{aligned}\omega_{\mathcal{C}\cap\mathcal{S}} &= (\omega_{\mathcal{S}} + \mathcal{C})\Big|_{\mathcal{C}\cap\mathcal{S}} = \omega_{\mathcal{S}}\Big|_{\mathcal{C}\cap\mathcal{S}} + (\varphi^*\mathcal{L}^2)\cdot\mathcal{S}\\ &= \varphi^*\omega_{\mathcal{L}}\cdot\mathcal{S} + \sum_{E_k\not\subset Y_0}\mu_k E_k\mathcal{S}\mathcal{C} + \sum_{E_k\subset\mathcal{T}}(\mu_k-1)E_k\mathcal{S}\mathcal{C}.\end{aligned} \tag{4.9}$$

Clearly, (4.9) gives us the ramification locus of the map $\varphi : \mathcal{C}\cap\mathcal{S} \to \mathcal{L}_0$ by Riemann-Hurwitz.

Let us consider the curve $\mathcal{C}\cap\mathcal{T}$. Since

$$\varphi_*(\mathcal{C}\cap\mathcal{T}) = \varphi_*(\varphi^*\mathcal{L}\cdot\mathcal{T}) = \mathcal{L}\cdot\varphi_*\mathcal{T} = 0, \tag{4.10}$$

φ contracts every component of $\mathcal{C}\cap\mathcal{T}$. In addition, since $\mathcal{C}$ is base point free, we see that

$$\mathcal{C}\cap E_k \ne \emptyset \text{ if and only if } \dim(\varphi(E_k)\cap Y_0) > 0. \tag{4.11}$$

We have

$$\begin{aligned}\omega_{\mathcal{C}\cap\mathcal{T}} &= (\omega_{\mathcal{T}} + \mathcal{C})\mathcal{C}\Big|_{\mathcal{T}} = \omega_{\mathcal{T}}\mathcal{C}\Big|_{\mathcal{T}} + \mathcal{C}^2\mathcal{T} = \omega_{\mathcal{T}}\mathcal{C}\Big|_{\mathcal{T}}\\ &= (\omega_{Y/\Delta}+\mathcal{T})\mathcal{T}\mathcal{C} = (\omega_{Y/\Delta}+\mathcal{T})Y_0\mathcal{C} - (\omega_{Y/\Delta}+\mathcal{T})\mathcal{S}\mathcal{C}\\ &= \sum_{E_k\not\subset Y_0}\mu_k E_k Y_0\mathcal{C} - \sum_{E_k\not\subset Y_0}\mu_k E_k\mathcal{S}\mathcal{C} - \sum_{E_k\subset\mathcal{T}}(\mu_k+1)E_k\mathcal{S}\mathcal{C}\\ &= \sum_{E_k\not\subset Y_0}\mu_k E_k\mathcal{T}\mathcal{C} - \sum_{E_k\subset\mathcal{T}}(\mu_k+1)E_k\mathcal{S}\mathcal{C}.\end{aligned} \tag{4.12}$$

Therefore,

$$\sum_{E_k\not\subset Y_0}\mu_k E_k\mathcal{T}\mathcal{C} = \omega_{\mathcal{C}\cap\mathcal{T}} + \sum_{E_k\subset\mathcal{T}}(\mu_k+1)E_k\mathcal{S}\mathcal{C}. \tag{4.13}$$

Let p be a point on $\mathcal{L}_0\backslash D$ and $U\subset\mathcal{L}$ be an analytic open neighborhood of $p\in\mathcal{L}$. Let $V\subset\mathcal{C}$ be a connected component of $\varphi^{-1}(U)$. We will show

that (4.8) holds when restrict to V. Since $\mathcal{L} \cap \varphi(E_k) \cap D = \emptyset$ for every $E_k \not\subset Y_0$, this is sufficient. If $V \cap \mathcal{T} = \emptyset$, the RHS of (4.8) vanishes and there is nothing to prove. Otherwise, $V \cap \mathcal{T}$ is a connected component of $\mathcal{C} \cap \mathcal{T}$. Let us assume

$$V \cap \mathcal{T} = \mathcal{C} \cap \mathcal{M} \tag{4.14}$$

where $\mathcal{M} \subset \mathcal{T}$ is an effective divisor contained in $\mathcal{T}$. Obviously, φ contracts $\mathcal{C} \cap \mathcal{M}$ to the point p.

Restricting (4.13) to $\mathcal{C} \cap \mathcal{M}$ yields

$$\sum_{E_k \not\subset Y_0} \mu_k E_k \mathcal{M}\mathcal{C} = \omega_{\mathcal{C}\cap\mathcal{M}} + \sum_{E_k \subset \mathcal{M}} (\mu_k + 1) E_k \mathcal{S}\mathcal{C}. \tag{4.15}$$

Let $\varphi_V : V \to U$ be the restriction of φ to V. By (4.9), when restricted to $V \cap \mathcal{S}$, $\varphi_{V\cap\mathcal{S}} : V \cap \mathcal{S} \to U_0$ is ramified along $E_k \cap \mathcal{S} \cap V$ with index $\mu_k + 1$ for $E_k \not\subset Y_0$ and with index μ_k if $E_k \subset \mathcal{M}$; the ramification indices at these points sum up to the degree of the map φ_V since φ_V contracts the components of V_0 other than those of $V \cap \mathcal{S}$, i.e., $\varphi_*(\mathcal{C} \cap \mathcal{M}) = 0$. Therefore,

$$\deg \varphi_V = \sum_{E_k \not\subset Y_0} (\mu_k + 1) E_k \mathcal{S}\mathcal{C} + \sum_{E_k \subset \mathcal{M}} \mu_k E_k \mathcal{S}\mathcal{C} \tag{4.16}$$

in V. On the other hand, when restricted to a general fiber, φ_V is ramified along $E_k \cap V_t$ with index $\mu_k + 1$ for each $E_k \not\subset Y_0$. Therefore, we have

$$\deg \varphi_V = \sum_{E_k \not\subset Y_0} (\mu_k+1) E_k \mathcal{S}\mathcal{C} + \sum_{E_k \subset \mathcal{M}} \mu_k E_k \mathcal{S}\mathcal{C} \ge \sum_{E_k \not\subset Y_0} (\mu_k+1) E_k Y_t \mathcal{C} \tag{4.17}$$

in V. Therefore,

$$\sum_{E_k \subset \mathcal{M}} \mu_k E_k \mathcal{S}\mathcal{C} \ge \sum_{E_k \not\subset Y_0} (\mu_k + 1) E_k \mathcal{M}\mathcal{C}. \tag{4.18}$$

Combining (4.15) and (4.18), we have

$$\sum_{E_k \not\subset Y_0} \mu_k E_k \mathcal{M}\mathcal{C} \ge \omega_{\mathcal{C}\cap\mathcal{M}} + \sum_{E_k \not\subset Y_0} (\mu_k + 1) E_k \mathcal{M}\mathcal{C} + \sum_{E_k \subset \mathcal{M}} E_k \mathcal{S}\mathcal{C} \tag{4.19}$$

and hence

$$\omega_{\mathcal{C}\cap\mathcal{M}} + \sum_{E_k \not\subset Y_0} E_k \mathcal{M}\mathcal{C} + \sum_{E_k \subset \mathcal{M}} E_k \mathcal{S}\mathcal{C} \le 0. \tag{4.20}$$

If $E_k \mathcal{M}\mathcal{C} = 0$ for all $E_k \not\subset Y_0$, there is nothing to prove. Otherwise, since

$$\deg \omega_{\mathcal{C}\cap\mathcal{M}} \ge -2, \sum_{E_k \not\subset Y_0} E_k \mathcal{M}\mathcal{C} \ge 1 \text{ and } \sum_{E_k \subset \mathcal{M}} E_k \mathcal{S}\mathcal{C} \ge 1, \tag{4.21}$$

the LHS of (4.20) is nonnegative and hence the equalities in (4.21) must all hold. Then (4.8) clearly follows from (4.15). □

Indeed, we have proved more than (4.2) in the above proof. In particular, since the equalities in (4.21) all hold, we see the following:

Proposition 4.2. *Let M be a connected component of $\mathcal{C} \cap \mathcal{T}$. Then*

$$\sum_{E_k \not\subset Y_0} E_k M \le 1 \tag{4.22}$$

and

$$M\mathcal{S} = 1. \tag{4.23}$$

In other words, M meets the union of horizontal exceptional divisors at no more than one point counted with multiplicity and it meets the rest of $\mathcal{C}_0$ at exactly one point.

Remark 4.3. Both Proposition 4.1 and 4.2 hold if we replace $(\varphi, \mathcal{L}, \mathcal{C})$ by $(\widehat{\varphi}, \widehat{\mathcal{L}}, \widehat{\mathcal{C}})$ (see (2.14), (2.31) and (2.32)), where $\widehat{\mathcal{L}}$ is a general member of

$$|\widehat{\mathcal{L}}_{\sigma_0,\sigma_1}| = |\sigma_0 \widehat{L} + \sigma_1 \widehat{R}_1| \tag{4.24}$$

and $\widehat{\mathcal{C}} = \widehat{\varphi}^* \widehat{\mathcal{L}}$. Here $\widehat{L}$ is the pullback of L under the map $\widehat{X} \to W$. Note that $\widehat{\mathcal{L}}_{\sigma_0,\sigma_1}$ is a very ample Cartier divisor on $\widehat{X}$ under our assumptions that $m|\sigma_1$ and $\sigma_0 >> \sigma_1 > 0$.

5. The Push-forward $\varepsilon_*(\varphi^*\mathcal{L})$

5.1. **Characterization of $\varepsilon_*(\mathcal{C} \cap \mathcal{T})$.** Now let us consider the push-forward $\varepsilon_* \mathcal{C}$. Obviously,

$$\varepsilon_* \mathcal{C}_0 = \varepsilon_*(\mathcal{C} \cap \mathcal{S}) + \varepsilon_*(\mathcal{C} \cap \mathcal{T}) \tag{5.1}$$

where every component of $\varepsilon(\mathcal{C} \cap \mathcal{T})$ is rational by (4.2). Indeed, $\mathcal{C} \cap \mathcal{T}$ is a disjoint union of trees of smooth rational curves and each connected component of $\mathcal{C} \cap \mathcal{T}$ meeting the rest of $\mathcal{C}_0$ at a single point by Proposition 4.1 and 4.2.

It is easy to see that the support $\operatorname{supp} \varepsilon_*(\mathcal{C} \cap \mathcal{T})$ of $\varepsilon_*(\mathcal{C} \cap \mathcal{T})$ is independent of the choices of $\mathcal{L}$ and $\mathcal{L}_{\sigma_0,\sigma_1}$: it is the support of the union of $\varepsilon_*(\varphi_T^{-1}(p))$ for all components $T \subset \mathcal{T}$ with $\dim(\varphi(T)) = 1$ and a general point $p \in \varphi(T)$, where $\varphi_T : T \to X$ is the restriction of φ to T.

Also we observe that since $\mathcal{C}$ is base point free and ε maps $\mathcal{S}$ birationally onto X_0', $\varepsilon_*(\mathcal{C} \cap \mathcal{S})$ is a linear system with base locus of dimension ≤ 0, i.e., consisting of isolated points, as $\mathcal{L}$ varies in $|\mathcal{L}_{\sigma_0,\sigma_1}|$. In other words, $\operatorname{supp} \varepsilon_*(\mathcal{C} \cap \mathcal{T})$, if nonempty, is the base locus $\operatorname{Bs}(\varepsilon_* \mathcal{C}_0)$ of $\varepsilon_* \mathcal{C}_0$ in dimension one. The base locus $\operatorname{Bs}(\varepsilon_* \mathcal{C})$ is independent of our choice of Y, the resolution of indeterminacy of the rational map $\varphi \circ \varepsilon^{-1} : X' \dashrightarrow X$. So $\operatorname{supp} \varepsilon_*(\mathcal{C} \cap \mathcal{T})$ is independent of not only the choices of $\mathcal{L}$ and $\mathcal{L}_{\sigma_0,\sigma_1}$ but also the choice of Y. Indeed, it is an invariant associated to the rational map $\varphi \circ \varepsilon^{-1}$.

As mentioned at the very beginning, $\varphi \circ \varepsilon^{-1} : X' \dashrightarrow X$ can be resolved by resolving the base locus of $\varepsilon_* \mathcal{C}$. So understanding $\varphi \circ \varepsilon^{-1}$ is more or less equivalent to understanding $\operatorname{Bs}(\varepsilon_* \mathcal{C})$. This shows the significance of $\varepsilon_*(\mathcal{C} \cap \mathcal{T})$. However, what makes $\varepsilon_*(\mathcal{C} \cap \mathcal{T})$ really important to us is the following observation.

Definition 5.1. For a rational map $g : A \dashrightarrow B$, we call $\mathrm{Exec}(g) \subset A$ the exceptional locus of g, which is the union of all curves in the set

$$\begin{aligned} \{C : \quad & C \subset A \text{ a reduced and irreducible curve,} \\ & g \text{ is regular at the generic point of } C \text{ and } g_*C = 0\} \end{aligned} \tag{5.2}$$

where the push-forward g_*C is the closure of the push-forward $g_*(C \cap U)$ with $U \subset A$ the open set over which g is regular.

Proposition 5.2. *Let $0 \le i \le m$ be an integer such $\varphi_* Q_i \ne 0$. Then*

$$\mathrm{Exec}(\varphi \circ \varepsilon_{Q_i}^{-1}) \subset \varepsilon_*(\mathcal{C} \cap \mathcal{T}) \tag{5.3}$$

and

$$\varphi\left(\mathrm{Exec}(\varphi \circ \varepsilon_{Q_i}^{-1})\right) \subset D, \tag{5.4}$$

where $\varepsilon_{Q_i} : Q_i \to P_i$ is the restriction of ε to Q_i. The same holds if we replace $(\varphi, \mathcal{C})$ by $(\widehat{\varphi}, \widehat{\mathcal{C}})$.

Proof. Let $G \subset P_i$ be a reduced and irreducible curve in $\mathrm{Exec}(\varphi \circ \varepsilon_{Q_i}^{-1})$ and $\Gamma = \varepsilon_{Q_i,*}^{-1}(G)$ be the proper transform of G under ε_{Q_i}. Then $\varphi_*\Gamma = 0$. It suffices to show that $G \subset \varepsilon_*(\mathcal{C} \cap \mathcal{T})$ and $\varphi(\Gamma) \subset D$.

Suppose that $\varphi(Q_i) = R_j$ for some $1 \le j \le 2$. For convenience, we write $Q = Q_i$, $P = P_i$, $R = R_j$, $\varphi_Q = \varphi_{Q_i}$ and $\varepsilon_Q = \varepsilon_{Q_i}$. Since $\varphi_*\Gamma = 0$, Γ is contained in the discriminant locus of the map $\varphi_Q : Q \to R$. That is, $\Gamma \subset \omega_Q - \varphi_Q^*\omega_R$. By (2.48),

$$\begin{aligned} \Gamma &\subset \omega_Q - \varphi_Q^*\omega_R \\ &= (\varphi_Q^* D - D_{i-1} - D_i) + (\omega_Q + D_{i-1} + D_i). \end{aligned} \tag{5.5}$$

Obviously, $\varepsilon_*(\omega_Q + D_{i-1} + D_i) = 0$. Therefore, $\Gamma \subset \varphi_Q^* D - D_{i-1} - D_i$ and hence $\Gamma \subset \varphi_Q^{-1}(D)$. It follows that $\varphi(\Gamma) \subset D$. That is, φ contracts Γ to a point on D.

It remains to show that $G \subset \varepsilon_*(\mathcal{C} \cap \mathcal{T})$. Let p be a general point G. It suffices to show that $\varphi_*(\varepsilon^{-1}(p)) \ne 0$.

We fix a sufficiently ample divisor B on X'. Let $B \in |B|$ be a general member passing through p. The pullback $A = \varepsilon^* B \subset Y$ is a flat family of curves over Δ passing through a point $q \in \varepsilon^{-1}(p) \cap \Gamma$. Let Σ be the connected component of $A \cap \mathcal{T}$ such that $q \in \Sigma$ and $\varepsilon(\Sigma) = p$. Obviously, Σ is supported on $\varepsilon^{-1}(p)$ and Σ meets $A \cap \mathcal{S}$ at the single point q. Let us consider the map $\varphi : A \to X$ locally at q. It maps q to the point $\varphi(q) = \varphi(\Gamma)$ lying on D and the component $A \cap Q_i$ to an irreducible curve on R_j passing through $\varphi(q)$. Obviously, $\varphi(A \cap Q_i) \ne D$ for B general. So we may apply Lemma 3.1 to conclude that $\varphi(\Sigma)$ contains an irreducible curve lying on R_{3-j} passing through $\varphi(q)$. It follows that $\varphi_*\Sigma \ne 0$ and hence $\varphi_*(\varepsilon^{-1}(p)) \ne 0$. □

Remark 5.3. The converse of (5.3), i.e.,

$$\operatorname{supp} \varepsilon_*(\mathcal{C} \cap \mathcal{T}) \cap P_i \subset \mathrm{Exec}(\varphi \circ \varepsilon_{Q_i}^{-1}) \tag{5.6}$$

also holds but is considerably harder to prove. We are not going to do it here since we have no use for it.

5.2. **Basic properties of $\varepsilon_*(\mathcal{C}\cap\mathcal{T})$.** We start with a few basic facts about a component of $\mathcal{C}_0$ not contracted by ε.

Proposition 5.4. *Let Γ be an irreducible component of $\mathcal{C}_0$ with $\varepsilon_*\Gamma \neq 0$. Suppose that $G = \varepsilon(\Gamma) \subset P_i$ for some $0 \le i \le m$. Then*

(1) *$\varepsilon : \Gamma \to X'$ is an immersion at every point $q \in \Gamma$ whose image $\varepsilon(q) \in P_{i-1} \cup P_{i+1}$, i.e., it induces an injection on the tangent spaces $T_{\Gamma,q} \hookrightarrow T_{X',\varepsilon(q)}$;*
(2) *at every point $q \in \Gamma$ whose image $\varepsilon(q) \in P_{i-1} \cup P_{i+1}$, there is a component $\Gamma' \subset \mathcal{C}_0$ with $q \in \Gamma'$, $\varepsilon_*(\Gamma') \neq 0$ and $\varepsilon(\Gamma') \subset P_{i-1} \cup P_{i+1}$;*
(3) *ε maps Γ birationally onto its image G if $\Gamma \subset \mathcal{S}$;*
(4) *$\Gamma = \mathcal{C} \cap Q_i$ if $\varphi_* Q_i \neq 0$ and $\Gamma \subset Q_i$.*
(5) *G is a fiber of P_i/D if $\Gamma \subset \mathcal{T}$ and $G \subset P_i$ for some $0 < i < m$.*

The same holds true if we replace $(\varphi, \mathcal{C})$ by $(\widehat{\varphi}, \widehat{\mathcal{C}})$.

Proof. By Lemma 3.2, $\varepsilon^* P_{i-1}$ and $\varepsilon^* P_{i+1}$ meet Γ transversely and (1) follows; (2) also follows directly from Lemma 3.2.

Since $\mathcal{C} \cap Q_i$ is base point free, ε maps $\Gamma \subset \mathcal{C} \cap Q_i$ birationally onto its image.

If $\varphi_* Q_i \neq 0$, $\mathcal{C}$ is big and base point free on Q_i. Therefore, $\mathcal{C} \cap Q_i$ is irreducible and $\Gamma = \mathcal{C} \cap Q_i$.

The last statement follows directly from the fact that Γ is rational if $\Gamma \subset \mathcal{C} \cap \mathcal{T}$. □

A connected component M of $\mathcal{C} \cap \mathcal{T}$ will meet $\mathcal{C} \cap \mathcal{S}$ at a single point. This fact leads to the following.

Proposition 5.5. *Let M be a connected component of $\mathcal{C} \cap \mathcal{T}$. We write*

$$M = M_0 + M_1 + ... + M_m \tag{5.7}$$

where $\varepsilon(M_i) \subset P_i$ for $0 \le i \le m$. Then for each $0 \le i < m$, either

$$\varepsilon_* M_i \cdot P_{i+1} = \varepsilon_* M_{i+1} \cdot P_i \tag{5.8}$$

or

$$\varepsilon_* M_i \cdot P_{i+1} = \varepsilon_* M_{i+1} \cdot P_i \pm p \tag{5.9}$$

for some $p \in P_i \cap P_{i+1}$ and (5.9) holds for at most one i. Or equivalently, we have either

$$f_* M_0 \cdot D = f_* M_m \cdot D \tag{5.10}$$

or

$$f_* M_0 \cdot D = f_* M_m \cdot D \pm p \tag{5.11}$$

for some $p \in D$, where the intersections are taken on R_1 and R_2, respectively. The same holds true if we replace $\mathcal{C}$ by $\widehat{\mathcal{C}}$.

Proof. If the intersection multiplicities $(\varepsilon_* M_i \cdot P_{i+1})_p$ and $(\varepsilon_* M_{i+1} \cdot P_i)_p$ do not agree at some point $p \in P_i \cap P_{i+1}$, say

$$\alpha = (\varepsilon_* M_i \cdot P_{i+1})_p - (\varepsilon_* M_{i+1} \cdot P_i)_p \neq 0, \tag{5.12}$$

then by Lemma 3.2, M will meet the rest of $\mathcal{C}_0$ at $|\alpha|$ distinct points q_i with $\varepsilon(q_i) = p$ for $1 \le i \le |\alpha|$. Therefore, we must have either (5.8) or (5.9) and (5.9) cannot hold for more than one i.

Since $\varepsilon_* M_i$ are supported on the fibers of P_i/D for $0 < i < m$, we see that (5.10) or (5.11) follows. $\square$

Next, we have the following key fact.

Proposition 5.6. *Every point $p \in f_*(\mathcal{C} \cap \mathcal{T}) \cap D$ lies in the image of*

$$\rho : \operatorname{Pic}(R_1) \oplus \operatorname{Pic}(R_2) \to \operatorname{Pic}(D) \tag{5.13}$$

in $\operatorname{Pic}(D)$*, where ρ is the map given in* (2.9)*. The same holds true if we replace $\mathcal{C}$ by $\widehat{\mathcal{C}}$.*

Proof. It is enough to show that $p \in \operatorname{Im}(\rho)$ for every point $p \in f_*(M) \cap D$ and every connected component M of $\mathcal{C} \cap \mathcal{T}$.

Note that M is a tree of smooth rational curves. We will prove it inductively by constructing a sequence of trees of smooth rational curves with marked points.

Let $q = M \cap \mathcal{S}$. We start with $M_0 = M$. Let Γ be a component of M_0 such that Γ has valence 1 in M_0 and $q \notin \Gamma$. Suppose that $G = \varepsilon(\Gamma) \subset P_i$.

If $G \cap (P_{i-1} \cup P_{i+1}) = \emptyset$, we simply remove Γ from M_0 and let $M_1 = M_0 - \Gamma$. Otherwise, since Γ has valence 1 in M_0, Γ meets $\varepsilon^*(P_{i-1} + P_{i+1})$ transversely at exactly one point q', where $q' = \Gamma \cap \Gamma'$ with Γ' a component of M.

It is clear that $f(q') = f_* \Gamma$ in $\operatorname{Pic}(D)$ and hence $f(q') \in \operatorname{Im}(\rho)$. Now we remove Γ from M_0 and let $M_1 = M_0 - \Gamma$ with one marked point q' on Γ'.

We continue this process to get a sequence $M_0, M_1, M_2, ..., M_n$ of trees of smooth rational curves with marked points. For each M_a and a component $\Gamma \subset M_a$ with $\varepsilon(\Gamma) \subset P_i$ for some i, we have

$$\Gamma \cdot \varepsilon^*(P_{i-1} + P_{i+1}) = \sum q_j + \sum r_k + q \tag{5.14}$$

if $q \in \Gamma$ and $\varepsilon(q) \in P_{i-1} \cup P_{i+1}$ and

$$\Gamma \cdot \varepsilon^*(P_{i-1} + P_{i+1}) = \sum q_j + \sum r_k \tag{5.15}$$

otherwise, where $\{q_j\}$ are all the marked points of M_a lying on Γ and $\{r_k\}$ are the intersections between Γ and $M_a - \Gamma$ satisfying $\varepsilon(r_k) \in P_{i-1} \cup P_{i+1}$.

Inductively, we have $f(q_j) \in \operatorname{Im}(\rho)$ for all marked points $q_j \in M_a$. We construct M_{a+1} by removing a component Γ of M_a which has valence 1 and does not contain q. Suppose that $q' = \Gamma \cap \Gamma'$ for a component Γ' of M_a. Then by (5.15), we have $f(q') \in \operatorname{Im}(\rho)$ if $\varepsilon(q') \in P_{i-1} \cup P_{i+1}$, where we assume that $\varepsilon(\Gamma) \subset P_i$.

If $\varepsilon(q') \in P_{i-1} \cup P_{i+1}$, we let $M_{a+1} = M_a - \Gamma$ with one extra marked point q'; otherwise, we simply let $M_{a+1} = M_a - \Gamma$.

Continue this process and we will eventually arrive at M_n, which consists of a single component Γ passing through q. By (5.15), $f(q) \in \operatorname{Im}(\rho)$ if $\varepsilon(q) \in P_{i-1} \cup P_{i+1}$ with $\varepsilon(\Gamma) \subset P_i$.

Clearly, a point $p \in f_*(M) \cap D$ is either $f(q)$ or $f(q')$ for a marked point q' on some M_a. Hence $p \in \operatorname{Im}(\rho)$ for all $p \in f_*(M) \cap D$. $\square$

5.3. **The case** $\alpha_0 = 1$**.** Now we are ready to handle the case $\alpha_0 = 1$. Namely, we will prove

Proposition 5.7. *If* $\alpha_0 = 1$*, then* $\deg \phi = 1$*.*

It suffices to prove $\deg \varphi_{Q_0} = \deg \varphi_{D_0} = 1$ by (2.49) and Corollary 2.10.

Proposition 5.8. *Let* $Q_i \subset \mathcal{S}$ *be a component of* Y_0 *with* $\varphi_* Q_i \neq 0$ *and let* p *be a point on* $P_i \cap (P_{i-1} \cup P_{i+1})$ *satisfying*

$$\dim \varphi(\varepsilon_{Q_i}^{-1}(p)) > 0 \tag{5.16}$$

where $\varepsilon_{Q_i} : Q_i \to P_i$ *is the restriction of* ε *to* Q_i*. Then* $p \in \varepsilon_*(\mathcal{C} \cap \mathcal{T})$*. More precisely,*

$$p = \varepsilon(\Gamma_1 \cap \Gamma_2) \tag{5.17}$$

for two components Γ_j *of* $\mathcal{C}_0$ *satisfying* $\varepsilon_* \Gamma_j \neq 0$*,* $\Gamma_1 \subset Q_i$ *and* $\Gamma_2 \subset \mathcal{T}$ *and hence*

$$p \in \varepsilon(\mathcal{C} \cap \mathcal{T} \cap Q_i). \tag{5.18}$$

The same holds true if we replace $(\varphi, \mathcal{C})$ *by* $(\widehat{\varphi}, \widehat{\mathcal{C}})$*.*

Proof. We write $P = P_i$, $Q = Q_i$, $\varphi_Q = \varphi_{Q_i}$ and $\varepsilon_Q = \varepsilon_{Q_i}$. The hypothesis (5.16) is equivalent to saying that the rational map $\varphi \circ \varepsilon_Q^{-1} : P \dashrightarrow X$ is not regular at p.

Let

$$\operatorname{Bs}(\varepsilon_*(\mathcal{C} \cap Q)) = \operatorname{Bs}(\varepsilon_* \varphi_Q^* \mathcal{L}) = \bigcap_{\mathcal{L} \in |\mathcal{L}_{\sigma_0, \sigma_1}|} \varepsilon_*(\mathcal{C} \cap Q) \tag{5.19}$$

be the base locus of $\varepsilon_*(\mathcal{C} \cap Q)$ as $\mathcal{L}$ varies in $|\mathcal{L}_{\sigma_0,\sigma_1}|$. Note that

$$\dim \operatorname{Bs}(\varepsilon_*(\mathcal{C} \cap Q)) \le 0. \tag{5.20}$$

It is easy to see that the map $\varphi \circ \varepsilon_Q^{-1}$ is not regular at p if and only if $p \in \operatorname{Bs}(\varepsilon_*(\mathcal{C} \cap Q))$. Therefore, $p \in \varepsilon_* \mathcal{C} = \varepsilon_* \varphi^* \mathcal{L}$ for all $\mathcal{L} \in |\mathcal{L}_{\sigma_0,\sigma_1}|$.

Suppose that (5.17) fails. Let $\Gamma_1 = \mathcal{C} \cap Q$. Since $p \in \operatorname{Bs}(\varepsilon_*(\mathcal{C} \cap Q))$, the component Γ_1 passes through p. Let $q \in \varepsilon^{-1}(p) \cap \Gamma_1$. Lemma 3.2 tells us there is a component Γ_2 of $\mathcal{C}_0$ such that $q \in \Gamma_1 \cap \Gamma_2$, $\varepsilon_* \Gamma_2 \neq 0$, $\varepsilon(\Gamma_2) \subset P_{i-1} \cup P_{i+1}$ and $\varepsilon_* \Gamma_2$ meets P_i transversely at p. Since (5.17) fails, $\Gamma_2 \subset \mathcal{C} \cap \mathcal{S}$ and hence $\Gamma_2 \subset Q_{i-1} \cup Q_{i+1}$. It follows that $q \in D_{i-1} \cup D_i$. Hence q is one of finitely many points on D_{i-1} and D_i that maps to p via ε. We conclude that $\mathcal{C} = \varphi^* \mathcal{L}$ has a base point at q. This is impossible since $\varphi^* \mathcal{L}$ is base point free as $\mathcal{L}$ varies in $|\mathcal{L}_{\sigma_0,\sigma_1}|$. $\square$

Corollary 5.9. *Let $Q_i \subset \mathcal{S}$ be a component of Y_0 with $\varphi_* Q_i \neq 0$. The rational map $\varphi \circ \varepsilon_{Q_i}^{-1}$ is regular and finite at a point p on $P_i \cap (P_{i-1} \cup P_{i+1})$ if $p \notin \varepsilon_*(\mathcal{C} \cap \mathcal{T})$. The same holds true if we replace $(\varphi, \mathcal{C})$ by $(\widehat{\varphi}, \widehat{\mathcal{C}})$.*

Proof. By the above proposition, $\varphi \circ \varepsilon_{Q_i}^{-1}$ is regular at p. It is also finite at p since $p \notin \operatorname{Exec}(\varphi \circ \varepsilon_{Q_i}^{-1})$ by Proposition 5.2. □

Proposition 5.10. *Let*

$$\delta_i = \varphi_{D_i} \circ f_{D_i}^{-1} : D \to D \tag{5.21}$$

with $f_{D_i} : D_i \to D$ the restriction of f to D_i for $0 \leq i \leq m-1$. Then

- $\delta_i^{-1}(\Lambda) \subset \operatorname{supp}(f_*(\mathcal{C} \cap \mathcal{T}) + f_*(\widehat{\mathcal{C}} \cap \mathcal{T}))$ *for $0 \leq i \leq m-1$. More precisely, if $\varphi_* Q_i \neq 0$, then*

$$\begin{aligned} \varepsilon(\varphi_{D_{i-1}}^{-1}(\Lambda)) \cup \varepsilon(\varphi_{D_i}^{-1}(\Lambda)) \subset \operatorname{Exec}(\varphi \circ \varepsilon_{Q_i}^{-1}) \cup \operatorname{Exec}(\widehat{\varphi} \circ \varepsilon_{Q_i}^{-1}) \\ \cup\, \varepsilon(\mathcal{C} \cap \mathcal{T} \cap Q_i) \cup \varepsilon(\widehat{\mathcal{C}} \cap \mathcal{T} \cap Q_i). \end{aligned} \tag{5.22}$$

- $\deg \varphi_{D_i} = \deg \delta_i = 1$ *for $0 \leq i \leq m-1$.*

Proof. When $\varphi_* Q_i = 0$, Q_i is contracted by φ along the fibers of Q_i/D. So $\delta_{i-1} = \delta_i$. Hence it suffices to prove the proposition for Q_i with $\varphi_* Q_i \neq 0$.

Let $Q = Q_i$. Suppose that $\varphi(Q) = R_1$. For a point $p \in \Lambda$, let Γ be a connected component of $\varphi_Q^{-1}(I_p)$. Obviously, $(D_{i-1} \cup D_i) \cap \Gamma \neq \emptyset$ and hence

$$\Big(\varphi_{D_{i-1}}^{-1}(p) \cup \varphi_{D_i}^{-1}(p)\Big) \cap \Gamma \neq 0. \tag{5.23}$$

On the other hand, it is clear that every point in $\varphi_{D_{i-1}}^{-1}(p) \cup \varphi_{D_i}^{-1}(p)$ lies on one of the connected components of $\varphi_Q^{-1}(I_p)$.

We observe that $\widehat{\varphi}_Q : Q \to \widehat{R}_1$ factors through $\varphi_Q : Q \to R_1$ with $R_1 \to \widehat{R}_1$ blowing down all I_p for $p \in \Lambda$. Therefore, $\widehat{\varphi}_* \Gamma = 0$.

Suppose that $\varepsilon_* \Gamma \neq 0$. Then

$$\varepsilon(\Gamma) \subset \operatorname{Exec}(\widehat{\varphi} \circ \varepsilon_{Q_i}^{-1}). \tag{5.24}$$

Suppose that $\varepsilon_* \Gamma = 0$. Let $q = \varepsilon(\Gamma)$. Since $I_p \subset \varphi(\Gamma)$ and $\Gamma \subset \varepsilon_Q^{-1}(q)$, $\varphi_* \varepsilon_Q^{-1}(q) \neq 0$. So $q \in \varepsilon(\mathcal{C} \cap \mathcal{T} \cap Q_i)$ by Proposition 5.8.

In conclusion, we have

$$\bigcup_{p \in \Lambda} \varepsilon(\varphi_Q^{-1}(I_p)) \subset \operatorname{Exec}(\widehat{\varphi} \circ \varepsilon_{Q_i}^{-1}) \cup \varepsilon(\mathcal{C} \cap \mathcal{T} \cap Q_i) \tag{5.25}$$

when $\varphi(Q) = R_1$. The same argument shows

$$\bigcup_{p \in \Lambda} \varepsilon(\widehat{\varphi}_Q^{-1}(\widehat{I}_p)) \subset \operatorname{Exec}(\varphi \circ \varepsilon_{Q_i}^{-1}) \cup \varepsilon(\widehat{\mathcal{C}} \cap \mathcal{T} \cap Q_i) \tag{5.26}$$

when $\varphi(Q) = R_2$, where $\widehat{I}_p$ is the exceptional curve of the blowup $\widehat{R}_2 \to R_2$ over $p \in \Lambda$. Therefore, (5.22) follows and

$$\delta_i^{-1}(\Lambda) \subset \operatorname{supp}(f_*(\mathcal{C} \cap \mathcal{T}) + f_*(\widehat{\mathcal{C}} \cap \mathcal{T})) \tag{5.27}$$

by Proposition 5.2. And by Proposition 5.6, every point $q \in \delta_i^{-1}(\Lambda)$ lies in the image $\mathrm{Im}(\rho)$ of the map ρ.

If $\deg \varphi_{D_i} = \deg \delta_i > 1$, then $\delta_i^{-1}(p)$ contains at least two distinct points $q_1 \neq q_2 \in \mathrm{Im}(\rho)$ for a point $p \in \Lambda$. By (1.5), $q_1 - q_2$ must be torsion. On the other hand, $\mathrm{Im}(\rho)$ is obviously torsion-free (see (2.9)). Contradiction and hence $\deg \delta_i = 1$. $\square$

Proposition 5.7 then follows easily. This settles the case $\alpha_0 = 1$.

5.4. **The case $\alpha_0 \neq 1$.** When $\alpha_0 \neq 1$, $\varphi_* Q_i \neq 0$ for some $1 \leq i \leq m-1$ by Corollary 2.11. Let $G \subset P_i$ be a general fiber of P_i/D. We see that the rational map $\varphi \circ \varepsilon_{Q_i}^{-1} : P_i \dashrightarrow R_j$ is regular along G. We make the key observation

$$G \cap \varepsilon_*(\mathcal{C} \cap \mathcal{T}) = \emptyset \Rightarrow G \cap \mathrm{Exec}(\varphi \circ \varepsilon_{Q_i}^{-1}) = \emptyset. \tag{5.28}$$

As a consequence, we see that $\varphi \circ \varepsilon_{Q_i}^{-1}$ maps G onto a curve Γ which only meets D at two points. This line of argument leads to the following.

Proposition 5.11. *Suppose that $\varphi(Q_i) = R_j$ for some $1 \leq i \leq m-1$ and $1 \leq j \leq 2$. Let $G \subset P_i$ be a general fiber of P_i/D, $\Gamma = \psi_{P_i}(G)$ and $\widehat{\Gamma} = \widehat{\psi}_{P_i}(G)$, where $\psi = \varphi \circ \varepsilon^{-1}$, $\widehat{\psi} = \widehat{\varphi} \circ \varepsilon^{-1}$ and $\psi_{P_i} = \varphi \circ \varepsilon_{Q_i}^{-1} : P_i \dashrightarrow R_j$ and $\widehat{\psi}_{P_i} = \widehat{\varphi} \circ \varepsilon_{Q_i}^{-1} : P_i \dashrightarrow \widehat{R}_j$ are the restrictions of ψ and $\widehat{\psi}$ to P_i, respectively. Then*

- *$\Gamma \subset R_j$ and $\widehat{\Gamma} \subset \widehat{R}_j$ are irreducible and base point free and meet D and $\widehat{D}$ set-theoretically at two points, respectively.*
- *$\Gamma \cdot D = \widehat{\Gamma} \cdot \widehat{D} \geq 2$.*
- *$\Gamma \cdot I_p = 0$ on R_j for all $p \in \Lambda$ if $j = 1$ and $\widehat{\Gamma} \cdot \widehat{I}_p = 0$ on $\widehat{R}_j$ for all $p \in \Lambda$ if $j = 2$.*
- *For all curves $A \subset \mathrm{Exec}(\phi_{R_j})$, $A \cdot \Gamma = 0$ on R_j, where ϕ_{R_j} is the restriction of ϕ to R_j.*
- *For all curves $A \subset \mathrm{Exec}(\xi^{-1} \circ \phi_{R_j})$, $A \cdot \Gamma = 0$ on R_j, where ξ is the birational map $\widehat{X} \dashrightarrow X$ in (2.31).*

Proof. We write $P = P_i$, $Q = Q_i$, $\psi_P = \psi_{P_i}$ and $\widehat{\psi}_P = \widehat{\psi}_{P_i}$.

Let $q_1 = G \cap P_{i+1}$ and $q_2 = G \cap P_{i-1}$. By (5.28), $\psi_P(q) \notin D$ for all $q \in G \backslash \{q_1, q_2\}$. Therefore, Γ meets D set-theoretically only at $\psi_P(q_1)$ and $\psi_P(q_2)$.

Clearly, Γ does not pass through a fixed point as G varies; otherwise, there is a curve $\Sigma \subset \mathrm{Exec}(\psi_P)$ with $\Sigma \cdot G \neq 0$. Therefore, $|\Gamma|$ is base point free. Similarly, $\widehat{\Gamma}$ meets $\widehat{D}$ set-theoretically at $\widehat{\psi}_P(q_1)$ and $\widehat{\psi}_P(q_2)$.

When $j = 1$, $\widehat{\psi}_P : P \dashrightarrow \widehat{R}_1$ factors through $\psi_P : P \dashrightarrow R_1$. If $\Gamma \cdot I_p \neq 0$, $p \in \widehat{\Gamma} \cap \widehat{D}$, while we have proved that $\widehat{\Gamma}$ and $\widehat{D}$ meet only at $\widehat{\psi}_P(q_1)$ and $\widehat{\psi}_P(q_2)$, which are general points on $\widehat{D}$ for G general. Therefore, we must have $\Gamma \cdot I_p = 0$ for all $p \in \Lambda$. Similarly, $\widehat{\Gamma} \cdot \widehat{I}_p = 0$ for all $p \in \Lambda$ when $j = 2$.

Suppose that there is a curve $A \subset \mathrm{Exec}(\phi_{R_j})$ such that $A \cdot \Gamma > 0$. Since Γ does not pass through a fixed point as G varies, Γ meets A at general points of A. Let us consider the rational map:

$$\phi \circ \psi : \quad \begin{tikzcd} X' \arrow[r, dashed, "\psi"] & X \arrow[r, dashed, "\phi"] & X \\ G \arrow[u, "\subset"'] \arrow[r] & \Gamma \arrow[u, "\subset"'] \end{tikzcd} \tag{5.29}$$

Applying the above argument to ϕ^2, we see that $\phi(\psi_P(G)) = \phi_{R_j}(\Gamma)$ is an irreducible curve on R_k meeting D only at 2 points, where we assume that $\phi(R_j) = R_k$ for some $1 \le k \le 2$. Since Γ meets A at general points of A, ϕ_{R_j} sends $\Gamma \cap A$ to some points in $\phi(A)$, which lie on D by Proposition 5.2. It follows that $\phi_{R_j}(\Gamma)$ meets D at points other than $\phi_{R_j}(\psi_P(q_1))$ and $\phi_{R_j}(\psi_P(q_2))$. Contradiction. Note that $\psi_P(q_1)$ and $\psi_P(q_2)$ are general points on D for G general. Therefore, $A \cdot \Gamma = 0$ for all $A \subset \mathrm{Exec}(\phi_{R_j})$. We can show the same statement for $A \subset \mathrm{Exec}(\xi^{-1} \circ \phi_{R_j})$ by considering

$$\xi^{-1} \circ \phi \circ \psi : \quad \begin{tikzcd} X' \arrow[r, dashed, "\psi"] & X \arrow[r, dashed, "\phi"] & X \arrow[r, dashed, "\xi^{-1}"] & \widehat{X} \\ G \arrow[u, "\subset"'] \arrow[r] & \Gamma \arrow[u, "\subset"'] \end{tikzcd} \tag{5.30}$$

□

Corollary 5.12. *Let $\Gamma_j \subset R_j$ be the union of $\psi_{P_i}(G)$ for all P_i satisfying $\psi(P_i) = R_j$.*

- *If $\Gamma_1^2 > 0$, then*

$$\mathrm{Exec}(\phi_{R_1}) \cup \mathrm{Exec}(\xi^{-1} \circ \phi_{R_1}) \subset \bigcup_{p \in \Lambda} I_p \cup C_1 \tag{5.31}$$

  where $C_1 = \emptyset$ when $g = 2$ or g is odd and $C_1 \subset R_1$ is the pullback of the (-1) curve on $S_1 \cong \mathbb{F}_1$ when $g \ge 4$ is even.
- *If $\Gamma_2^2 > 0$, then*

$$\mathrm{Exec}(\phi_{R_2}) \cup \mathrm{Exec}(\xi^{-1} \circ \phi_{R_2}) \subset C_2 \tag{5.32}$$

  where $C_2 = \emptyset$ when $g = 2$ or g is odd and $C_2 \subset R_2$ is the (-1) curve on $R_2 \cong \mathbb{F}_1$ when $g \ge 4$ is even.

Proof. This follows directly from Proposition 5.11. □

If

$$\bigcup_{i=1}^{m-1} \varphi(Q_i) = R_1 \cup R_2 \tag{5.33}$$

and $\Gamma_j^2 > 0$ for $j = 1, 2$, then both (5.31) and (5.32) hold by Corollary 5.12, which is the ideal situation for us. It is not hard to see that (5.33) is true if and only if one of the following two conditions holds:

- $\alpha_0 \neq 1$ (or equivalently $\deg \phi_{R_1} < \deg \phi$) and $\phi(R_1) \neq \phi(R_2)$; we can put this into the form

$$R_1 + R_2 \subset \phi_*(R_1) + \phi_*(R_2) \neq (\deg \phi)(R_1 + R_2). \tag{5.34}$$

- We have

$$\deg \phi_{R_1} + \deg \phi_{R_2} < \deg \phi. \tag{5.35}$$

Even if both (5.34) and (5.35) fail, we can still make (5.33) happen by replacing ϕ by ϕ^2.

Proposition 5.13. *One of the following must be true:*

(1) $\deg \phi_{R_1} = \deg \phi$.
(2) (5.34) *holds.*
(3) (5.35) *holds.*
(4) *We have*

$$\deg(\phi^2)_{R_1} + \deg(\phi^2)_{R_2} < \deg \phi^2 = (\deg \phi)^2 \tag{5.36}$$

where $(\phi^2)_{R_i} = \phi \circ \phi_{R_i}$ *are the restrictions of* ϕ^2 *to* R_i *for* $i = 1, 2$.

Proof. Suppose that all (1)-(3) fail. Then $\phi(R_1) = \phi(R_2) = R_j$ for some $1 \leq j \leq 2$ and $\deg \phi_{R_1} + \deg \phi_{R_2} = \deg \phi$. Then

$$\begin{aligned} \deg(\phi^2)_{R_1} + \deg(\phi^2)_{R_2} &= \deg(\phi_{R_j} \circ \phi_{R_1}) + \deg(\phi_{R_j} \circ \phi_{R_2}) \\ &= (\deg \phi_{R_j})(\deg \phi_{R_1} + \deg \phi_{R_2}) \\ &= (\deg \phi_{R_j})(\deg \phi) < (\deg \phi)^2. \end{aligned} \tag{5.37}$$

□

So we can always assume (5.33) when $\alpha_0 \neq 1$. The more serious issue is that we might have $\Gamma_j^2 = 0$ even if $\Gamma_j \neq 0$. This can be worked around with the following trick.

Proposition 5.14. *Assuming* (5.33)*, if* $\phi(R_j) = R_j$*, then there is an irreducible curve* $\Gamma \subset R_j$ *such that* Γ *is big and nef on* R_j*,* $\Gamma \cdot I_p = 0$ *for all* $p \in \Lambda$ *if* $j = 1$ *and*

$$A \cdot \phi^k(\Gamma) = 0 \tag{5.38}$$

for all $k \in \mathbb{Z}_{\geq 0}$ *and curves* $A \subset \operatorname{Exec}(\phi_{R_j}) \cup \operatorname{Exec}(\xi^{-1} \circ \phi_{R_j})$*. Consequently, we have* (5.31) *if* $j = 1$ *and* (5.32) *if* $j = 2$.

Proof. If $\Gamma_j^2 > 0$, then we are done. Otherwise, $\Gamma_j^2 = 0$ and hence $\Gamma_j = nH$ for some integer $n > 0$, where $H \cdot D = 2$ and H gives a ruling of R_j, i.e., $|H|$ is a pencil giving a map $R_j \to \mathbb{P}^1$. Let $H \in |H|$ be a general member of the pencil $|H|$. Note that $H = \psi_{P_i}(G)$ for some P_i with $\psi(P_i) = R_j$ and a general fiber G of P_i/D.

Since $A \cdot \Gamma_j = 0$ for all curves $A \subset \operatorname{Exec}(\phi_{R_j}) \cup \operatorname{Exec}(\xi^{-1} \circ \phi_{R_j})$,

$$H \cap \big(\operatorname{Exec}(\phi_{R_j}) \cup \operatorname{Exec}(\xi^{-1} \circ \phi_{R_j})\big) = \emptyset. \tag{5.39}$$

Therefore, $\Gamma = \phi_{R_j,*}(H)$ meets D at two points with multiplicities μ each, where $\mu = \alpha_0$ if $j = 1$ and $\mu = \beta_{m-1}$ if $j = 2$.

Note that $\phi_{R_j} \circ \eta_P = \psi_P$, where $P = P_0$ if $j = 1$ and $P = P_m$ if $j = 2$. So we may identify ϕ_{R_j} and ψ_P via the isomorphism $\eta_P : P \cong R_j$.

The key here is to prove that Γ is reduced and hence $\Gamma^2 > 0$ since $\mu > 1$. Now we take H to be a member of $|H|$ tangent to D at a point p, i.e., $H = 2p$ in $\mathrm{Pic}(D)$. It is not hard to see that $p \notin \mathrm{Im}(\rho)$. Therefore, $p \notin f_*(\mathcal{C} \cap \mathcal{T})$. Hence ϕ_{R_j} is regular and finite locally at p by Corollary 5.9. That is, ϕ_{R_j} is totally ramified along D at p with ramification index μ. It is easy to see that ϕ_{R_j} locally maps H at p to a smooth curve Γ tangent to D at $\phi_{R_j}(p)$ with multiplicity 2μ. This implies that Γ is reduced for H general and hence $\Gamma^2 > 0$ and Γ is big.

Similarly, $\widehat{\Gamma} = \xi_*^{-1}(\Gamma) = \xi_*^{-1}(\phi_{R_j,*}(H))$ is big and nef and meets $\widehat{D}$ at two points with multiplicities μ each. So $\Gamma \cdot I_p = 0$ for all $p \in \Lambda$ if $j = 1$ by the same argument in the proof of Proposition 5.11.

Finally, using the same argument in the proof of Proposition 5.11 again, we can conclude (5.38) by iterating ϕ and considering the maps

$$\phi^{k+2} \circ \psi : \quad \begin{tikzcd} X' \arrow[r, dashed, "\psi"] & X \arrow[r, dashed, "\phi"] & X \arrow[r, dashed, "\phi^k"] & X \arrow[r, dashed, "\phi"] & X \\ G \arrow[u, "\subset"'] \arrow[r] & H \arrow[u, "\subset"'] \arrow[r] & \Gamma \arrow[u, "\subset"'] \end{tikzcd} \tag{5.40}$$

and

$$\xi^{-1} \circ \phi^{k+2} \circ \psi : \quad \begin{tikzcd} X' \arrow[r, dashed, "\psi"] & X \arrow[r, dashed, "\phi"] & X \arrow[r, dashed, "\phi^k"] & X \arrow[r, dashed, "\xi^{-1}\circ\phi"] & \widehat{X} \\ G \arrow[u, "\subset"'] \arrow[r] & H \arrow[u, "\subset"'] \arrow[r] & \Gamma \arrow[u, "\subset"'] \end{tikzcd} \tag{5.41}$$

□

Note that we can always assume that $\phi(R_j) = R_j$ for some $1 \le j \le 2$; otherwise, if $\phi(R_1) = R_2$ and $\phi(R_2) = R_1$, we simply replace ϕ by ϕ^2.

Now we are ready to prove our main theorem when $\alpha_0 \ne 1$. We always assume (5.33).

Case $\phi(R_2) = R_2$ and g odd or $g = 2$. So we have (5.32) by Proposition 5.14. Here $R_2 \cong \mathbb{P}^2$ or $\mathbb{F}_0$ and hence

$$\mathrm{Exec}(\phi_{R_2}) = \emptyset. \tag{5.42}$$

Therefore, for a general member H of an ample linear system $|H|$ on R_2, $\phi_{R_2}(H)$ is a curve meeting D at $H \cdot D$ points with multiplicity $\beta = \beta_{m-1}$ each.

Suppose that there is a point $q \in R_2$ with

$$\dim(\varphi(f_Q^{-1}(q))) > 0 \tag{5.43}$$

where $f_Q : Q \to R_2$ is the restriction of f to $Q = Q_m$. That is, ϕ_{R_2} is not regular at q. Let $M = f_Q^{-1}(q)$.

We write

$$\varphi_Q^* D = \beta D_{m-1} + A \tag{5.44}$$

where $\varphi_* A = 0$. By (5.42), we necessarily have $f_* A = 0$. Since every connected component of $\operatorname{supp} A$ must meet D_{m-1}, $f(\operatorname{supp} A) \subset D$ and $f(A_1) \neq f(A_2)$ for any two distinct connected components A_1 and A_2 of $\operatorname{supp} A$. As a consequence, we see that $q \in D$ and $\varphi(M)$ meets D at the unique point $\delta(q)$, where $\delta = \delta_{m-1} : D \to D$ is the map defined in (5.21). Hence

$$\delta(bq) = \varphi_* M \tag{5.45}$$

in $\operatorname{Pic}(D)$ for some integer $b > 0$. Here we use δ for both the map $\delta : D \to D$ and the push-forward $\delta_* : \operatorname{Pic}(D) \to \operatorname{Pic}(D)$ induced by δ.

In addition, since $q \in D$,

$$q \in f_*(\mathcal{C} \cap \mathcal{T}) \cap D \subset \operatorname{Im}(\rho) \tag{5.46}$$

by Proposition 5.8.

When $g = 2$, $R_2 \cong \mathbb{P}^2$ and we choose H to be the hyperplane divisor. Then

$$\phi_{R_2,*}(H) = \beta H \Rightarrow \delta(\beta H) = \beta H \tag{5.47}$$

in $\operatorname{Pic}(D)$. Since $\varphi_* M$ and H are linearly dependent in $\operatorname{Pic}(R_2)$, we derive

$$\delta(3b\beta q - b\beta H) = 0 \Rightarrow b\beta H = 3b\beta q \tag{5.48}$$

by combining (5.45) and (5.47) and making use of (1.5) and the fact that $\deg \delta = 1$ by Proposition 5.10. Since $q \in \operatorname{Im}(\rho)$ and $\operatorname{Im}(\rho)$ is torsion free, $H = 3q$ by (5.48). It is not hard to see that such q cannot lie in $\operatorname{Im}(\rho)$. Contradiction. Therefore, ϕ_{R_2} is regular and finite everywhere. So we have

$$\phi_{R_2}^* D = \beta D \Rightarrow \beta = \beta^2 \Rightarrow \beta = 1 \tag{5.49}$$

and we are done.

When g is odd, $R_2 \cong \mathbb{F}_0$ and we let H_1 and H_2 be the two rulings of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Let

$$\phi_{R_2,*}(H_1) = a_{11} H_1 + a_{12} H_2 \text{ and } \phi_{R_2,*}(H_2) = a_{21} H_1 + a_{22} H_2 \tag{5.50}$$

in $\operatorname{Pic}(R_2)$ for some integers a_{ij}. Then the previous argument shows that

$$\delta(\beta H_1) = a_{11} H_1 + a_{12} H_2 \text{ and } \delta(\beta H_2) = a_{21} H_1 + a_{22} H_2 \tag{5.51}$$

in $\operatorname{Pic}(D)$. It follows that

$$\pm \beta(H_1 - H_2) = \delta(\beta H_1 - \beta H_2) = (a_{11} - a_{21}) H_1 - (a_{22} - a_{12}) H_2 \tag{5.52}$$

and hence

$$a_{11} - a_{21} = a_{22} - a_{12} = \pm\beta \tag{5.53}$$

where we make use of (1.5) and the fact that $\deg \delta = 1$ again.

The effectiveness of $\phi_{R_2,*}(H_i)$ implies that $a_{ij} \geq 0$. And

$$a_{11} + a_{12} = a_{21} + a_{22} = \beta. \tag{5.54}$$

This can only happen if either

$$\phi_{R_2,*}(H_1) = \beta H_1 \text{ and } \phi_{R_2,*}(H_2) = \beta H_2 \tag{5.55}$$

or

$$\phi_{R_2,*}(H_1) = \beta H_2 \text{ and } \phi_{R_2,*}(H_2) = \beta H_1. \tag{5.56}$$

In either case, ϕ_{R_2} maps a general member $H_1 \in |H_1|$ onto a curve $\Gamma \in |H_1|$ or $|H_2|$ with a map of degree β. That is, $\phi_{R_2,*}(H_1) = \beta\Gamma$.

Let H_1 be a member of the pencil $|H_1|$ tangent to D at a point p. We still have $\varphi_*(f_Q^* H_1) = \beta\Gamma$. On the other hand, $H_1 = 2p$ in $\mathrm{Pic}(D)$ and hence $p \notin \mathrm{Im}(\rho)$ and $p \notin f_*(\mathcal{C} \cap \mathcal{T})$. So ϕ_{R_2} is regular and finite at p. Hence $\Gamma = \phi_{R_2}(G)$ must be smooth and tangent to D at $\delta(p)$ with multiplicity 2β. Contradiction.

Case $\phi(R_2) = R_2$ and $g \geq 4$ even. Here $R_2 \cong \mathbb{F}_1$ and

$$\mathrm{Exec}(\phi_{R_2}) \subset E = C_2, \tag{5.57}$$

where E is the (-1)-curve on R_2.

Suppose that $\mathrm{Exec}(\phi_{R_2}) = E$. Then

$$\varphi_*(f_Q^* H) = \phi_{R_2,*}(H) = \beta H \Rightarrow \delta(\beta H) = \beta H \tag{5.58}$$

by Proposition 5.14, where H is the pullback of the hyperplane divisor under the blowup $\mathbb{F}_1 \to \mathbb{P}^2$ and β and δ are defined as above. Let

$$\varphi_*(f_Q^* E) = b_1 H + b_2 E \tag{5.59}$$

in $\mathrm{Pic}(R_2)$ for some integers b_i. For a general member $G \in |H - E|$, $\phi_{R_2,*}(G)$ meets D at three points $\delta(q_1), \delta(q_2)$ and $\delta(p)$, where $G \cdot D = q_1 + q_2$ and $p = E \cap D$; it meets D at $\delta(q_1)$ and $\delta(q_2)$ with multiplicity β each. Therefore,

$$\delta(\beta(H - E)) + \delta(b_3 p) = (\beta - b_1)H - b_2 E \Rightarrow \delta((\beta - b_3)E) = b_1 H + b_2 E \tag{5.60}$$

in $\mathrm{Pic}(D)$ for some integer $b_3 > 0$. Note that $\beta - b_3 > 0$ due to the effectiveness of $\varphi_*(f_Q^* E)$.

Combining (5.58) and (5.60) and arguing as before, we obtain

$$\pm\left((\beta - b_3)H - 3(\beta - b_3)E\right) = (\beta - b_3 - 3b_1)\, H - 3b_2 E \tag{5.61}$$

in $\mathrm{Pic}(R_2)$. Obviously, $b_2 \geq 0$ since $\varphi_*(f_Q^*(H - E))$ is nef. Therefore, (5.61) gives us $b_1 = 0$, $b_2 = \beta - b_3$,

$$\varphi_*(f_Q^* E) = b_2 E \text{ and } \delta(b_2 E) = b_2 E. \tag{5.62}$$

Let $q \in D$ be a point where ϕ_{R_2} is not regular and let $M = f_Q^{-1}(q)$. Then (5.62) shows that $\varphi(M) = E$ for $q \in E$.

Suppose that $q \notin E$. Using the previous argument we can show that $q \in D$ and $\varphi(M)$ meets D only at $\delta(q)$ and hence (5.45) holds.

Note that $\varphi_*(f_Q^* H) - \varphi_* M = \beta H - \varphi_* M$ is nef. Therefore, we have either $\varphi_* M \cdot E = 0$ or $E \subset \varphi(M)$.

If $\varphi_* M \cdot E = 0$, $\varphi_* M$ is a multiple of H and we can argue as in the case $R_2 \cong \mathbb{P}^2$ to show that $H = 3q$ using (5.45) and (5.58), which is impossible for $q \in \mathrm{Im}(\rho)$. Therefore, $E \subset \varphi(M)$. And since $\varphi(M)$ meets D at a single point, we must have $\varphi(M) = E$.

In conclusion, $\varphi(f_Q^{-1}(q)) = E$ for all points $q \in R_2$ where ϕ_{R_2} is not regular. Consequently, the rational map $g \circ \phi_{R_2} : R_2 \dashrightarrow \mathbb{P}^2$ is regular, where $g : R_2 \to \mathbb{P}^2$ is the blow-down of E. Let $D' = g(D)$. Then it is easy to see that

$$(g \circ \phi_{R_2})^* D' = \beta D + \beta E \Rightarrow \beta = \beta^2 \tag{5.63}$$

and we are done.

Suppose that $\mathrm{Exec}(\phi_{R_2}) = \emptyset$. Let

$$\varphi_*(f_Q^* H) = a_{11} H + a_{12} E \text{ and } \varphi_*(f_Q^* E) = a_{21} H + a_{22} E \tag{5.64}$$

in $\mathrm{Pic}(R_2)$ for some integers a_{ij}. Then

$$\delta(\beta H) = a_{11} H + a_{12} E \text{ and } \delta(\beta E) = a_{21} H + a_{22} E \tag{5.65}$$

in $\mathrm{Pic}(D)$. By the same argument as in the case $R_2 \cong \mathbb{F}_0$, we see that

$$\pm \beta(H - 3E) = (a_{11} - 3a_{21}) H - (3a_{22} - a_{12}) E \tag{5.66}$$

and hence

$$3(a_{11} - 3a_{21}) = 3a_{22} - a_{12} = \pm 3\beta. \tag{5.67}$$

Combining with $3a_{11} + a_{12} = 3\beta$, we obtain

$$a_{11} = a, a_{12} = 3\beta - 3a, a_{21} = \frac{a}{3} \mp \frac{\beta}{3} \text{ and } a_{22} = \pm\beta + \beta - a. \tag{5.68}$$

Obviously, $\varphi_*(f_Q^* H)$ is big and nef. Hence $a_{11} > |a_{12}|$, $a_{12} \le 0$ and

$$\beta \le a < \frac{3}{2}\beta. \tag{5.69}$$

The effectiveness of $\varphi_*(f_Q^* E)$ requires that $a_{21} + a_{22} \ge 0$. Therefore,

$$a_{21} = \frac{a}{3} - \frac{\beta}{3} \text{ and } a_{22} = 2\beta - a. \tag{5.70}$$

The nefness of $\varphi_*(f_Q^*(H - E))$ requires that $a_{11} - a_{21} \ge a_{22} - a_{12}$. Hence we must have $a = \beta$,

$$\varphi_*(f_Q^* H) = \beta H \text{ and } \varphi_*(f_Q^* E) = \beta E. \tag{5.71}$$

Note that this implies

$$\phi_{R_2,*}(H - E) = \varphi_*(f_Q^*(H - E)) = \beta(H - E) \tag{5.72}$$

which means that ϕ_{R_2} maps a general member G of the pencil $|H - E|$ onto $\Gamma = \phi_{R_2}(G) \in |H - E|$ with a map of degree β. Using the same argument as in the case $R_2 \cong \mathbb{F}_0$, we can show that this is impossible.

Case $\phi(R_1) = R_1$. Now we simply switch from $(X, \varphi, \phi, \varepsilon)$ to $(\widehat{X}, \widehat{\varphi}, \widehat{\phi}, \widehat{\varepsilon})$ (see (2.31) and (2.32)). Obviously, $\widehat{\phi}(\widehat{R}_1) = \widehat{R}_1$ and $\widehat{R}_1 \cong \mathbb{P}^2$, $\mathbb{F}_0$ or $\mathbb{F}_1$. This reduces it to the previous cases.

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